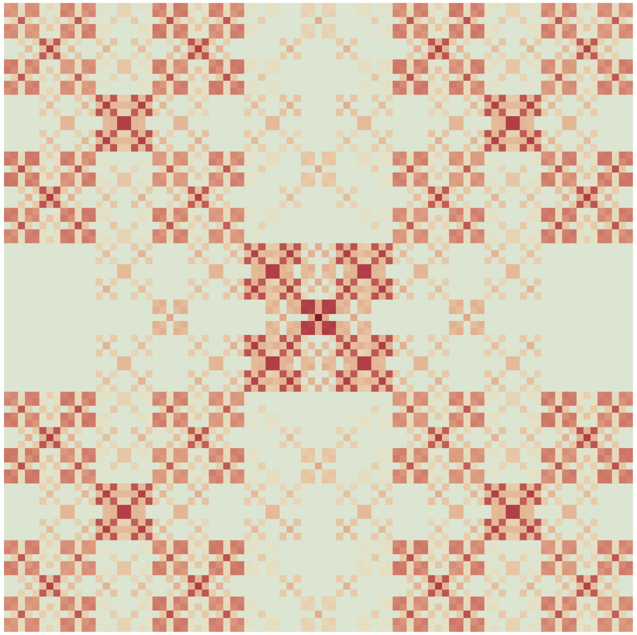
Multifractal Properties of the Fibonacci Chain

Frédéric Piéchon
Laboratoire de Physique des Solides, Orsay
CNRS, Université Paris-Sud, France

Nicolas Macé



Anu Jagannathan

Waves in random media :

P.W. Anderson 1958
« gang of four », 1981
scaling theory of metal-insulator transition

- Weak disorder and d>2:
  - -extended states
  - -normal diffusive motion of wave packet
  - -gaussian distribution of conductance

Metallic regime

- Strong disorder or d<2:
  - -exponentially localized states
  - -no diffusion of wave packet
  - -lognormal distribution of conductance

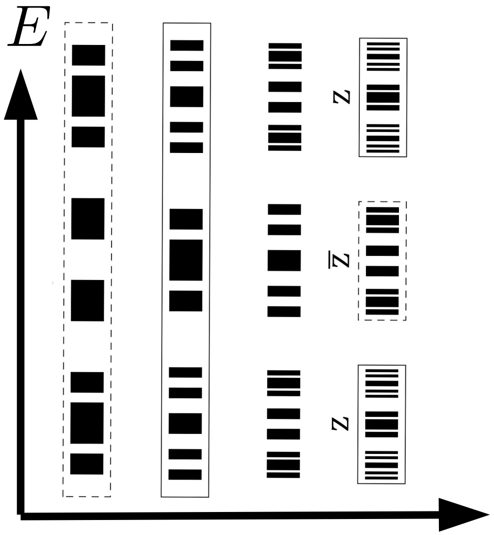
Insulating regime

- At the transition :
  - -critical states with multifractal properties
  - -anomalous diffusion of wave packet
  - -anomalous power law tail of conductance distribution

Critical regime at the MIT

Waves in quasiperiodic chains:

Fractal spectrum



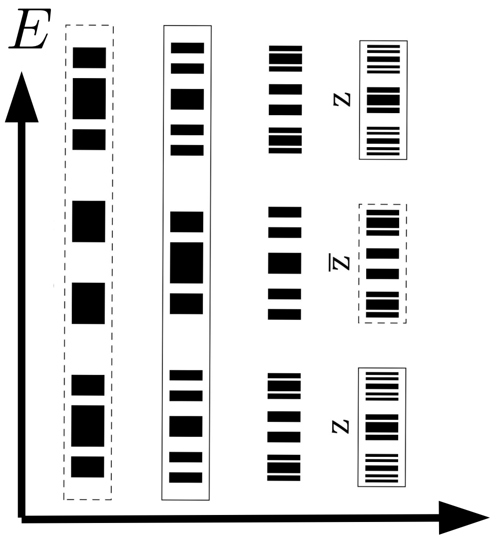
Period of approximant

Fractal dimensions:

$$D_q$$

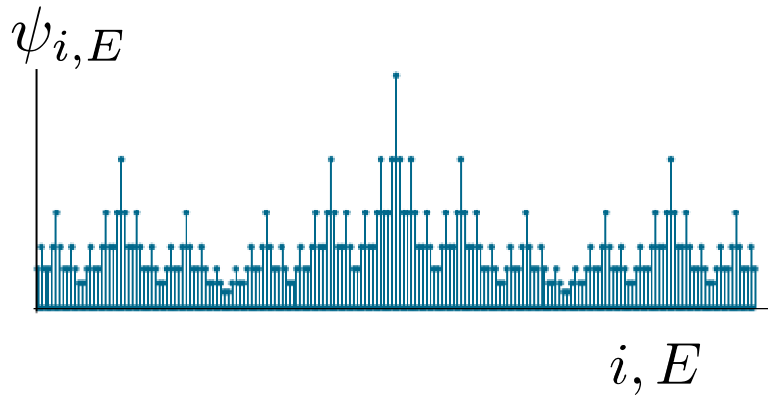
Waves in quasiperiodic chains:

Fractal spectrum



Period of approximant

Self similar Wave functions



Fractal dimensions:

$$D_q$$

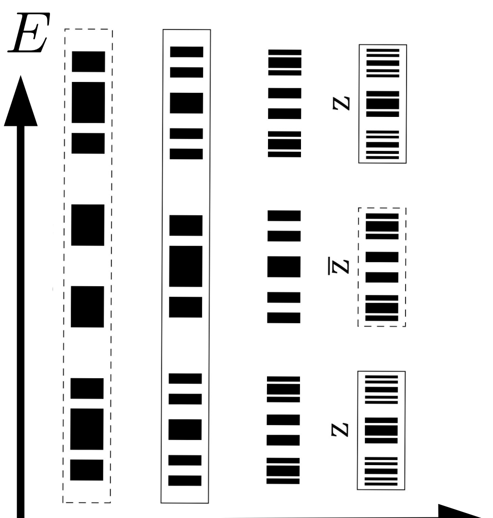
Wave functions: $D_q^{\psi}(E)$

$\tilde{D}_q^{\psi}(i)$

Local Density of states : $D_q^{\mu}(i)$

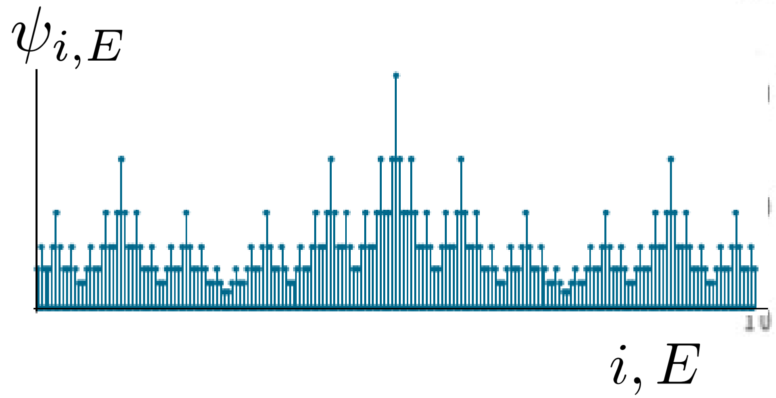
Waves in quasiperiodic chains:

Fractal spectrum

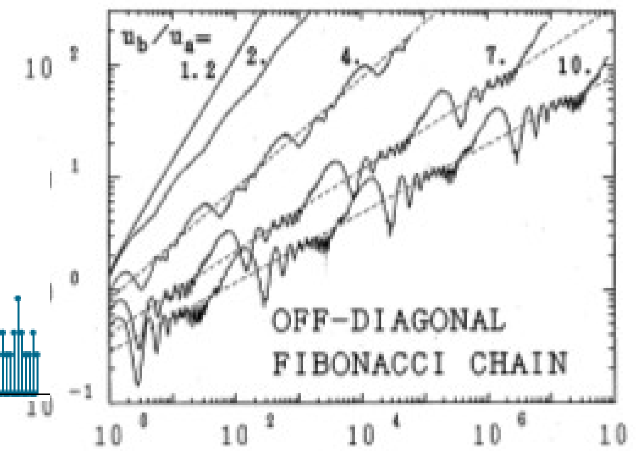


Period of approximant

Self similar Wave functions



Anomalous diffusion of wavepacket



$$\langle x^q \rangle \sim t^{q\sigma_q}$$

Fractal dimensions:

$$D_q$$

Wave functions: $D_q^{\psi}(E)$ $\tilde{D}_q^{\psi}(i)$

Local Density of states : $D_q^{\mu}(i)$

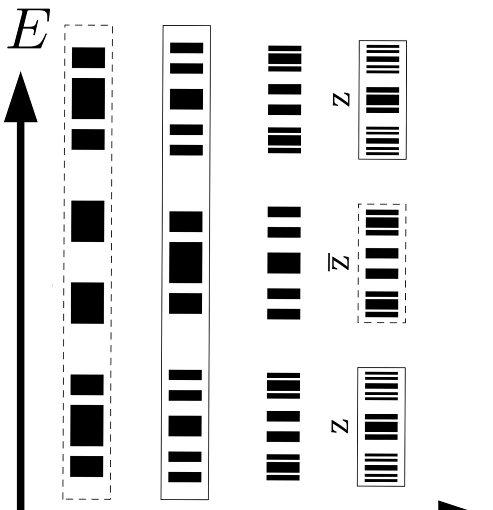
Diffusion exponents :

$$\sigma_q(i)$$

Fractal properties are generic and tunable

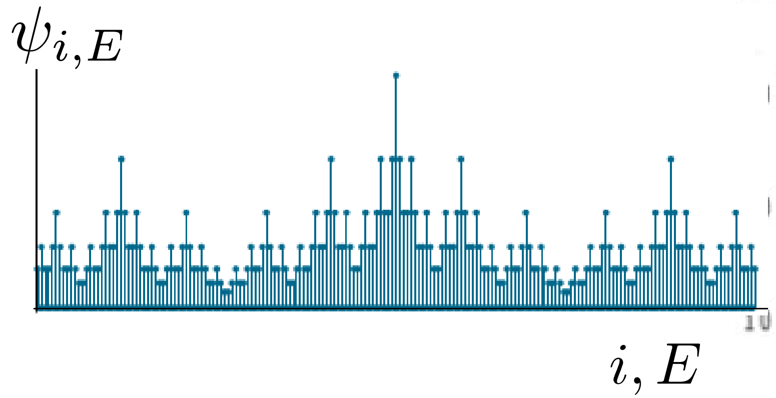
Waves in quasiperiodic chains:

Fractal spectrum



Period of approximant

Self similar Wave functions



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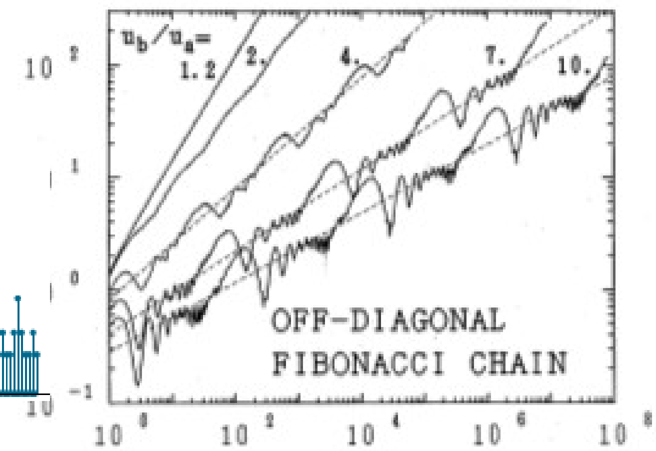

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Local Density of states : $D_q^{\mu}(i)$

Diffusion exponents :

$$\sigma_q(i)$$

How to relate spectrum fractal dimensions
with wave functions fractal dimensions and diffusion exponents ?

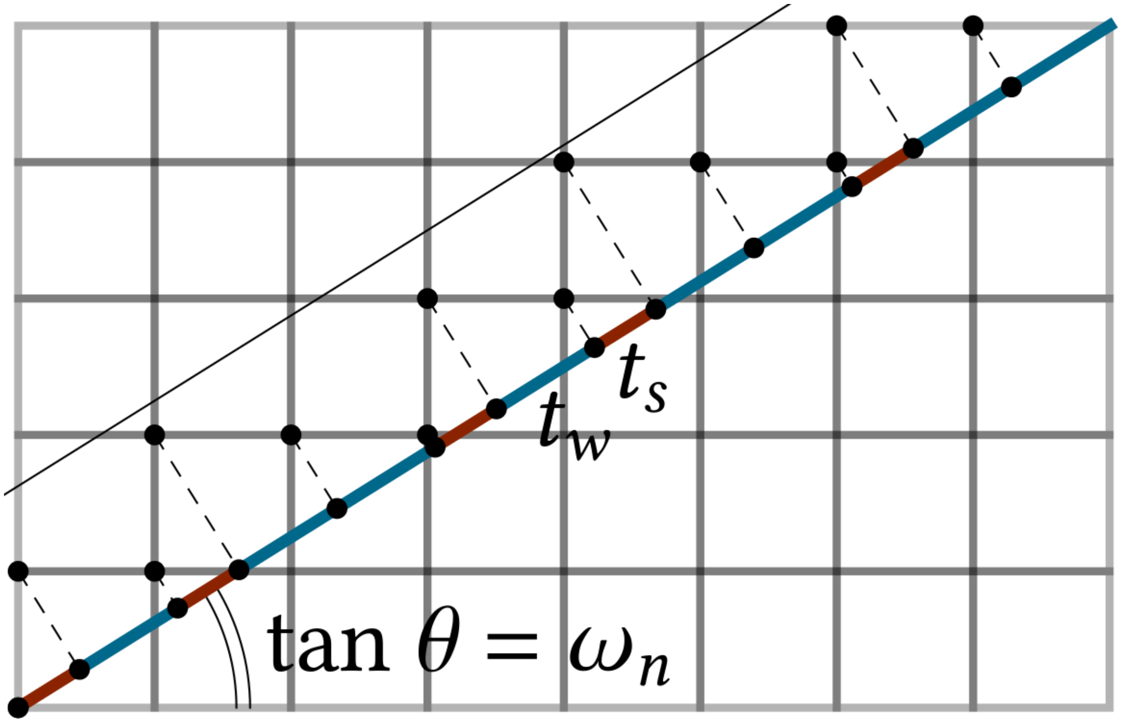
PART 1

Tight-binding model on the Fibonacci chain

Perturbative renormalization group : *atoms & molecules*

Energy spectrum fractal properties

Tight-binding model on the Fibonacci chain



$$F_{n+1} = F_n + F_{n-1}$$

$$\omega_n = \frac{F_{n-1}}{F_n}$$

$$\omega_n \to \omega = \frac{\sqrt{5}-1}{2}$$

« golden mean »

Atoms sites: F_{n-3}

$$E\ \psi_i = t_w(\psi_{i-1} + \psi_{i+1})$$

Molecules sites : $2 \times F_{n-2}$

$$E\ \psi_i = t_w\ \psi_{i\pm1} + t_s\ \psi_{i\mp1}$$

$$\rho = \frac{t_w}{t_s} \leq 1$$

$\rho \to 1$ weak modulation

$\rho \ll 1$ strong modulation

Atoms and molecules :

Niu & Nori (1986)
Kalugin,Kitaev,Levitov (1986)

$t_w = 0$

Atoms : isolated sites F_{n-3}

$$E\ \psi_i = t_w \cancel{(\psi_{i-1} + \psi_{i+1})}$$

Molecules : isolated dimer F_{n-2}

$$E\ \psi_i = t_w\ \psi_{i\pm 1} + \cancel{t_s\ \psi_{i\mp 1}}$$

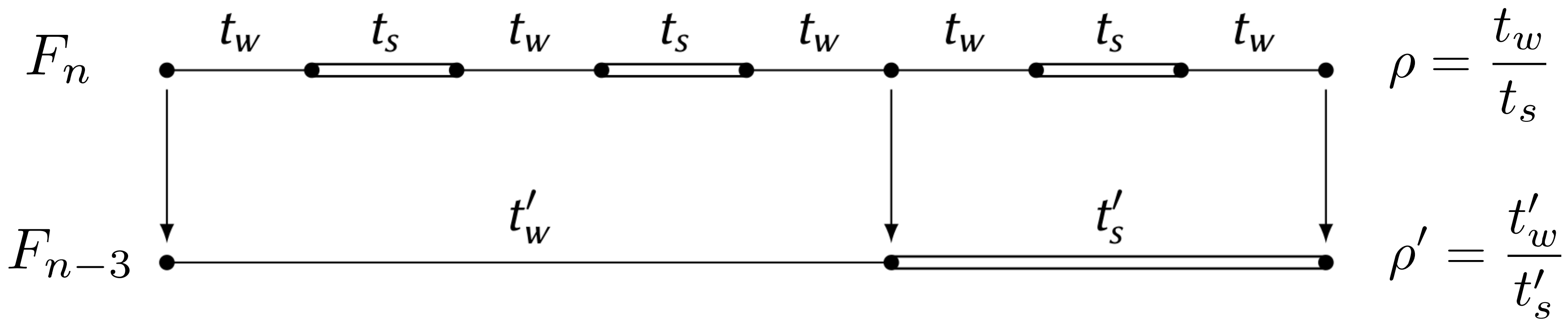
Energy level	Degeneracy
$E = 0$ (central)	F_{n-3}
$E = \pm t_s$ Bonding/antibonding (lateral)	F_{n-2}

Perturbative renormalization group

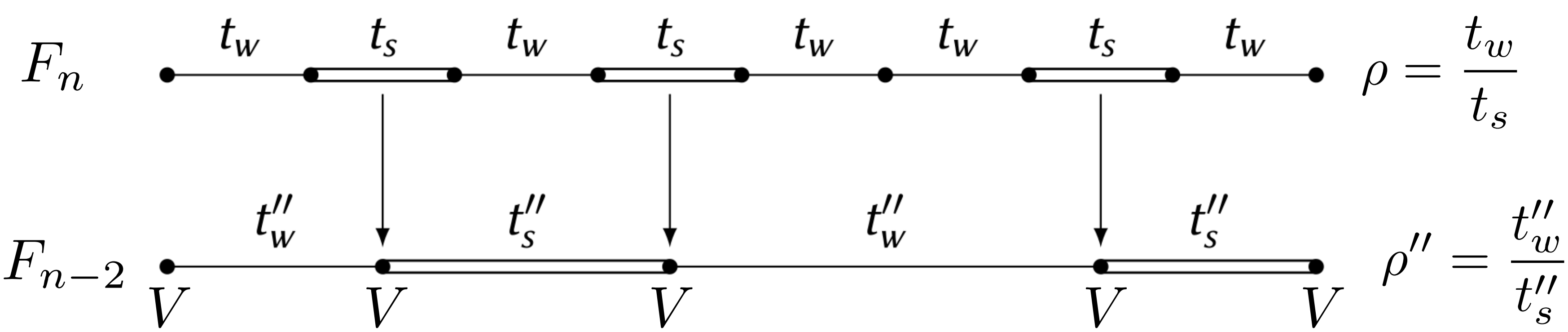
Niu & Nori (1986)
Kalugin,Kitaev,Levitov (1986)

$$\rho = t_w/t_S \ll 1$$

Atomic RG step : (decimation of molecules)



Molecular RG step: (decimation of atoms)



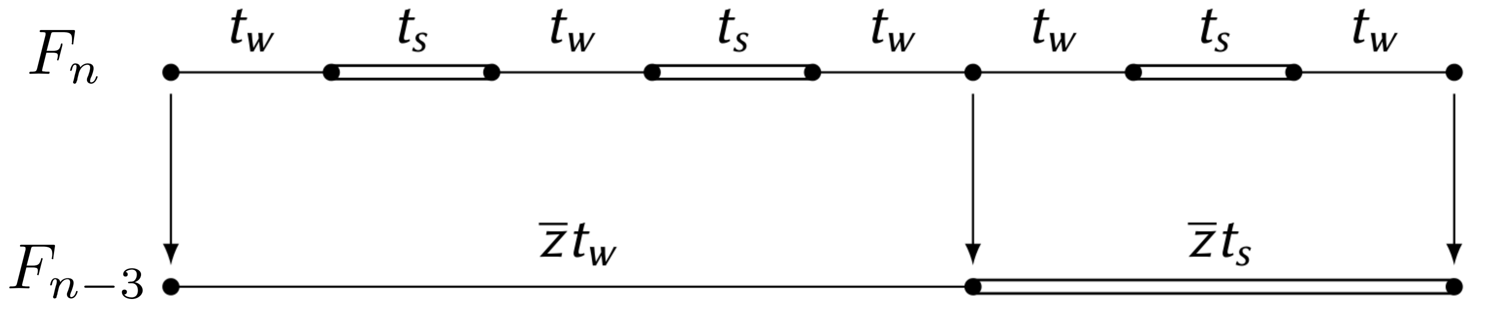
$V = \pm t_s$ (bonding/antibonding)

Perturbative renormalization group

Niu & Nori (1986)
Zheng (1987)
Piéchon, Benakli, Jagannathan (1995)

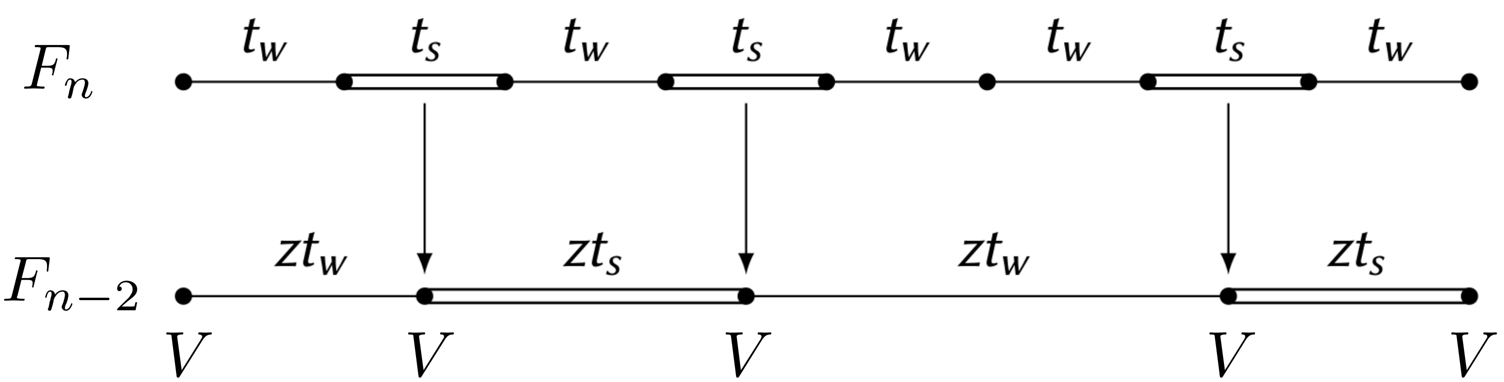
Atomic RG step : (decimation of molecules)

$$\rho = \frac{t_w}{t_s} \ll 1$$



$$\bar{z} \simeq \rho^2$$

Molecular RG step: (decimation of atoms)



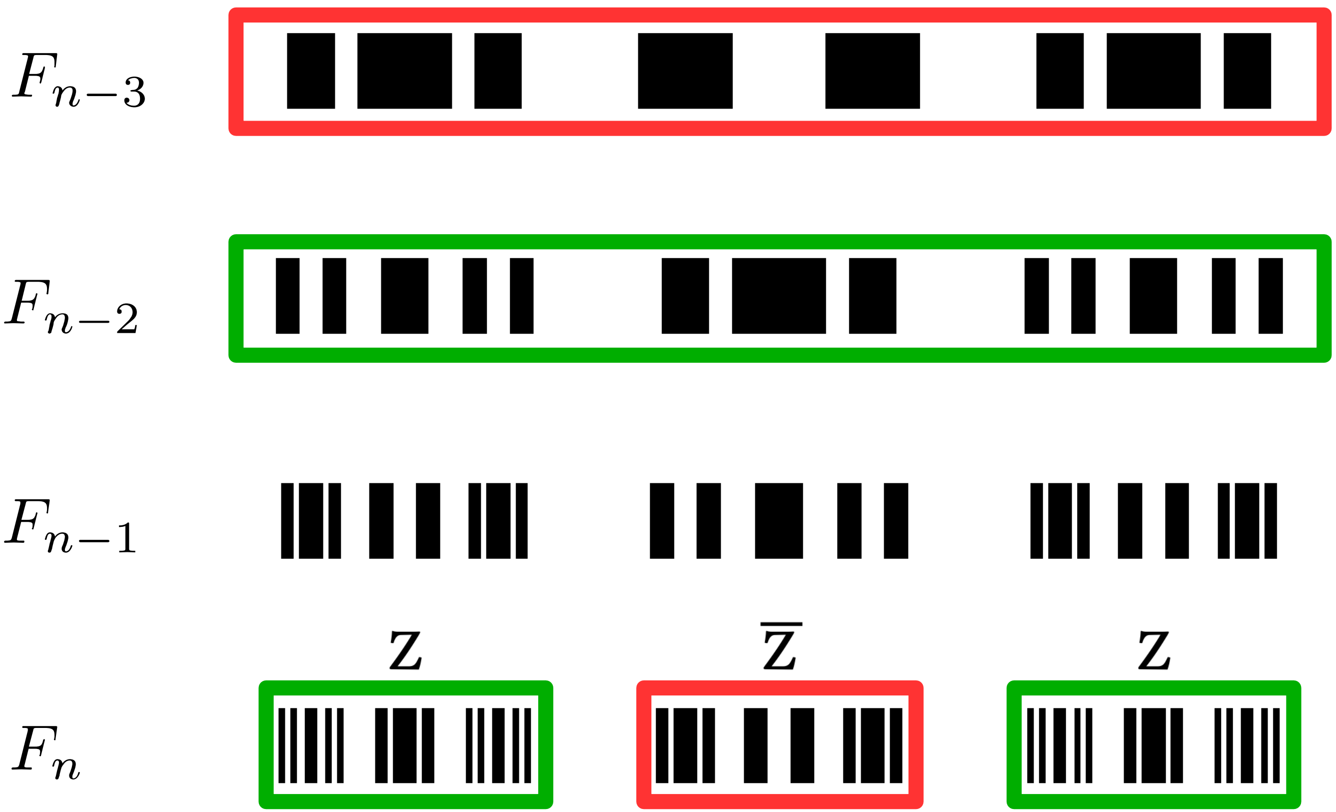
$$z = \rho/2$$

$V = \pm t_s$ (bonding/antibonding)

RG reconstruction of energy band spectrum

Niu & Nori (1986)
Zheng (1987)
Piéchon et al (1995)

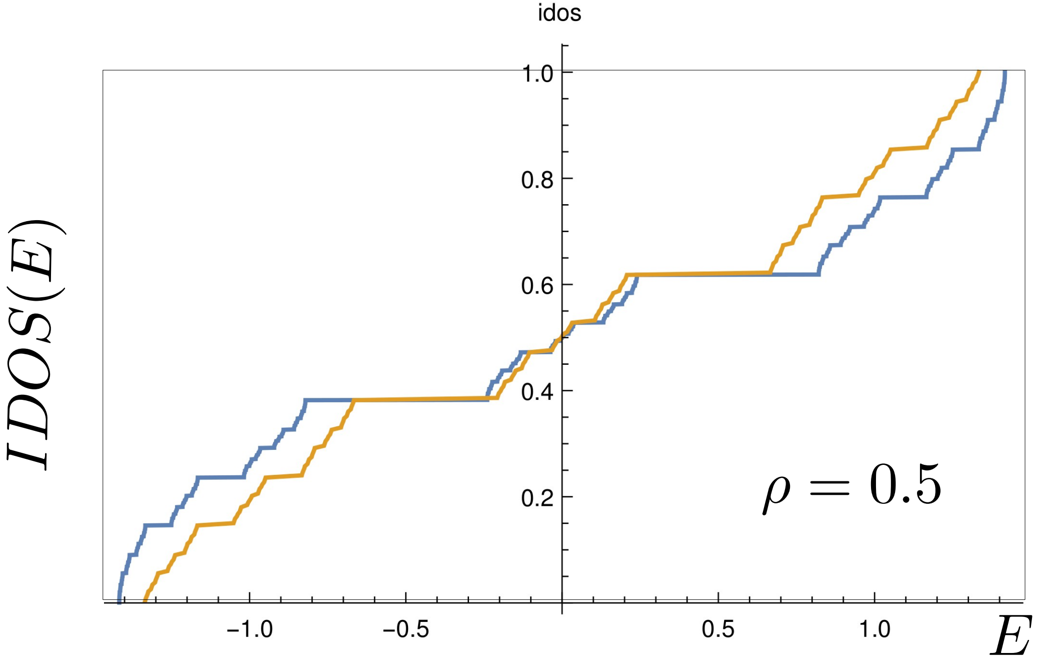
$$H_n = \underbrace{(-t_s + \mathrm{z}H_{n-2})}_{\text{bonding levels}} \oplus \underbrace{(\bar{\mathrm{z}}H_{n-3})}_{\text{atomic levels}} \oplus \underbrace{(t_s + \mathrm{z}H_{n-2})}_{\text{antibonding levels}}$$



Density of states :

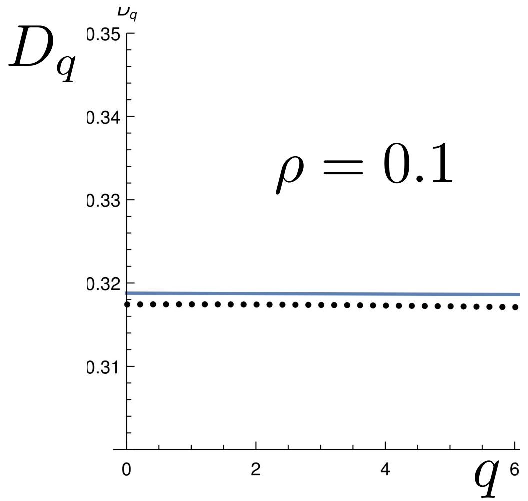
$$d\mu_n(E) = \omega^2 d\mu_{n-2}(\mathrm{z}E - t_s) + \omega^3 d\mu_{n-3}(\bar{\mathrm{z}}E) + \omega^2 d\mu_{n-2}(\mathrm{z}E + t_s)$$

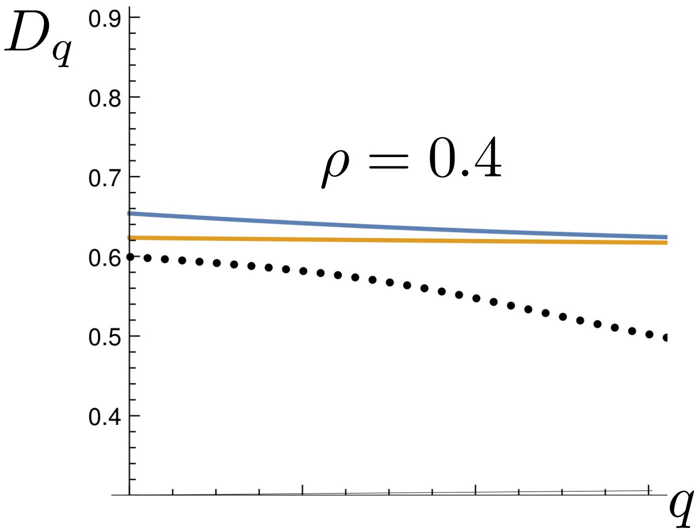
Recursive reconstruction of the IDOS :



Fractal dimensions:

$$2\omega^{2q}\mathrm{z}^{(1-q)D_q} + \omega^{3q}\bar{\mathrm{z}}^{(1-q)D_q} = 1$$





Nonperturbative formula : (trace map)

$$I = (\rho - 1/\rho)^2/4 \qquad J = \frac{1}{8}(3 + \sqrt{25 + 16I})$$

Kohmoto,Sutherland,Tang (1987)

(6 cycle) $\bar{\mathrm{z}}(\rho) = \sqrt{(4(I+1)^2 + 1} - 2(I+1)$

(2 cycle) $\mathrm{z}_{\mathrm{edge}}(\rho) = \frac{1}{2}(8J - 1 - \sqrt{(8J-1)^2 - 4})$

(4 cycle) $\mathrm{z}(\rho) = \sqrt{\frac{16 + 7I - \sqrt{(16+7I)^2 - 4}}{2}}$

$\bar{\mathrm{z}}(1) = \omega^3$

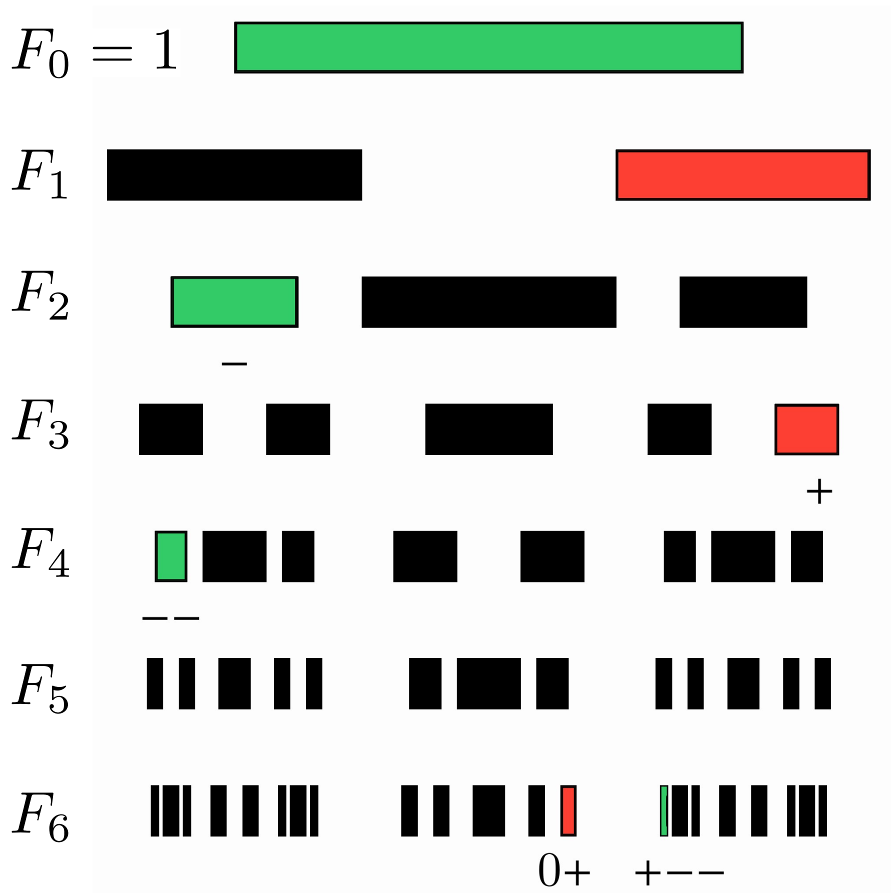
$\mathrm{z}_{\mathrm{edge}}(1) = \omega^4$ (Van-hove singularity)

$\mathrm{z}(1) = \omega^2$

Rüdinger,Piéchon (1998)

RG path of energy bands

bonding RG step (left) → $-$
atomic RG step (central) → 0
antibonding RG step (right) → $+$



Each energy band is labelled by a sequence of three indices :

central band : $(00...0)$
bottom band: $(- - ...-)$

generic band of chain F_n

$$\Delta_n(E) \rightarrow \mathcal{S}_E = (.. - .. + ..0..)$$

$$2(n_- + n_+) + 3n_0 = n$$

$$\Delta_n(E) = \mathrm{z}^{(n_+ + n_-)} \bar{\mathrm{z}}^{n_0}$$

$$N(\Delta) = 2^{n_+ + n_-} \frac{(n_+ + n_- + n_0)!}{n_0!(n_+ + n_-)!}$$

Bands fractal properties

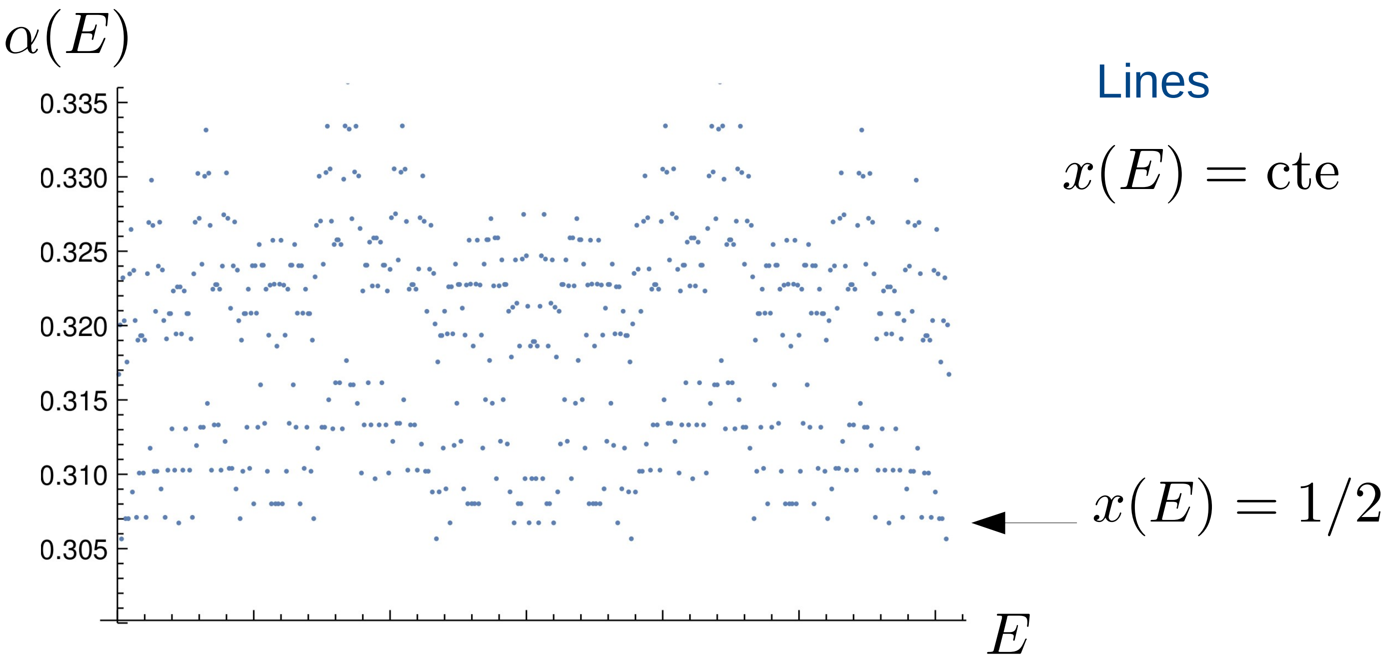
Bands anomalous scaling :

$$\begin{cases} \Delta_n(E) \sim F_n^{-1/\alpha(E)} \\ N(\Delta) \sim F_n^{g(\alpha)} \end{cases}$$

$$1/\alpha(x(E)) = \frac{1-2x}{3}\frac{\log \bar{\mathrm{z}}}{\log \omega} + x\frac{\log \mathrm{z}}{\log \omega}$$

$$x(E) = \frac{n_+ + n_-}{n}$$

$$g(\alpha(x)) = \frac{x\log(3x/2) - (1+x)\log(1+x)^{1/3} + (1-2x)\log(1-2x)^{1/3}}{\log \omega}$$

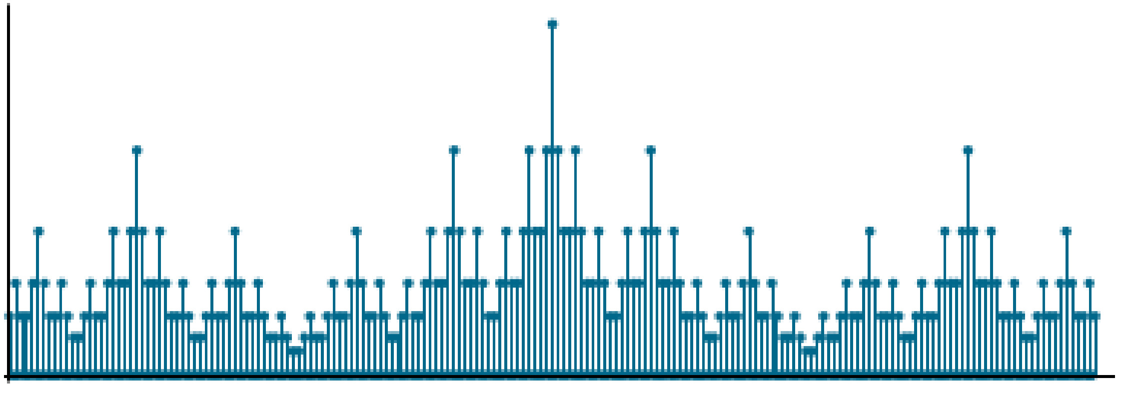


Lines

$x(E) = \mathrm{cte}$

PART 2

What about the wave functions ?



Fractal properties of wave functions

The aim of the game :

$$\begin{cases} \sum_i |\psi_n(i,E)|^{2q} = F_n^{(1-q)D_q^{\psi}(E)} \\ \sum_E |\psi_n(i,E)|^{2q} = F_n^{(1-q)\tilde{D}_q^{\psi}(i)} \\ d\mu(i,E) \longrightarrow D_q^{\mu}(i) \end{cases}$$

Local spectral measure

What is known :

-Exact wave function fractal properties for central level
Kohmoto,Sutherland,Tang (1987)

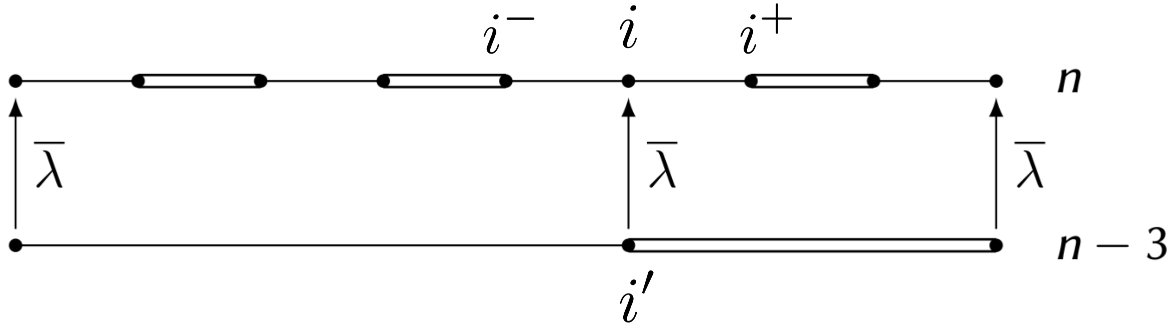
-RG and fractal properties to zero order in ρ
Piéchon (1996), Thiem, Schreiber (2013)

Lets go beyond

RG step for Wave functions

Atomic RG step :

$\rho \ll 1$



$\bar{\lambda} \simeq 1/(1+\rho^2)$

For E in the central cluster :

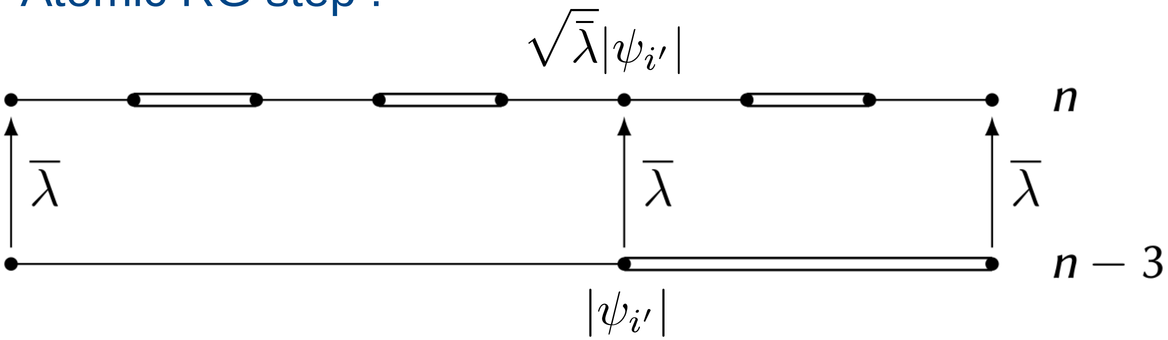
atom site i : $|\psi_n(i,E)|^2 = \bar{\lambda}|\psi_{n-3}(i',E')|^2$

molecule site $i^{\pm}$: $|\psi_n(i^{\pm},E)|^2 \simeq \rho^2\bar{\lambda}|\psi_{n-3}(i',E')|^2$

RG step for Wave functions

Atomic RG step :

$\rho \ll 1$

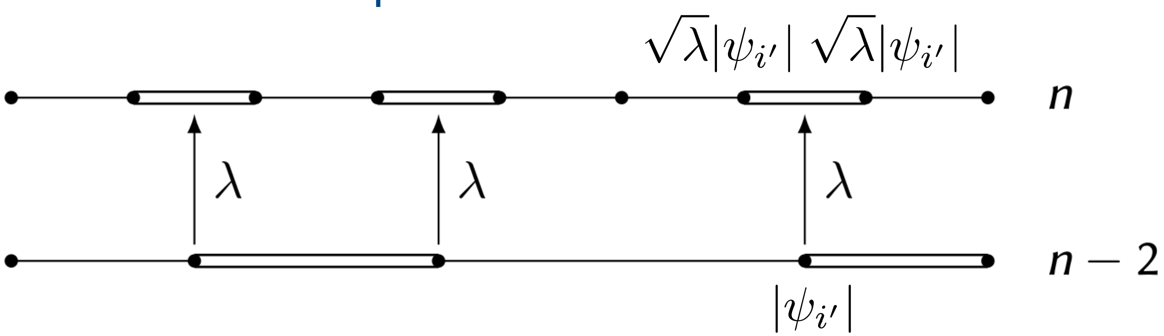


$$\bar\lambda \simeq 1/(1+\rho^2)$$

$$|\psi_n(i,E)|^2 = \bar\lambda|\psi_{n-3}(i',E')|^2$$

if i is an atom site and E in the central cluster

Molecular RG step :



$$\bar\lambda \simeq 1/(2+\rho^2)$$

$$|\psi_n(i,E)|^2 = \lambda|\psi_{n-2}(i',E')|^2$$

if i is a molecule site and E in the lateral clusters

Fractal Dimensions $D_q^\psi(E)$

$$\sum_i |\psi_n(i,E)|^{2q} = F_n^{(1-q)D_q^\psi(E)}$$

RG step for sites with largest probability : $|\psi_n(i,E)|^{2q} = \begin{cases} \bar\lambda^q |\psi_{n-3}(i',E')|^{2q} \\ \lambda^q |\psi_{n-2}(i',E')|^{2q} \end{cases}$

For a an energy E with renormalzation path $x(E)$

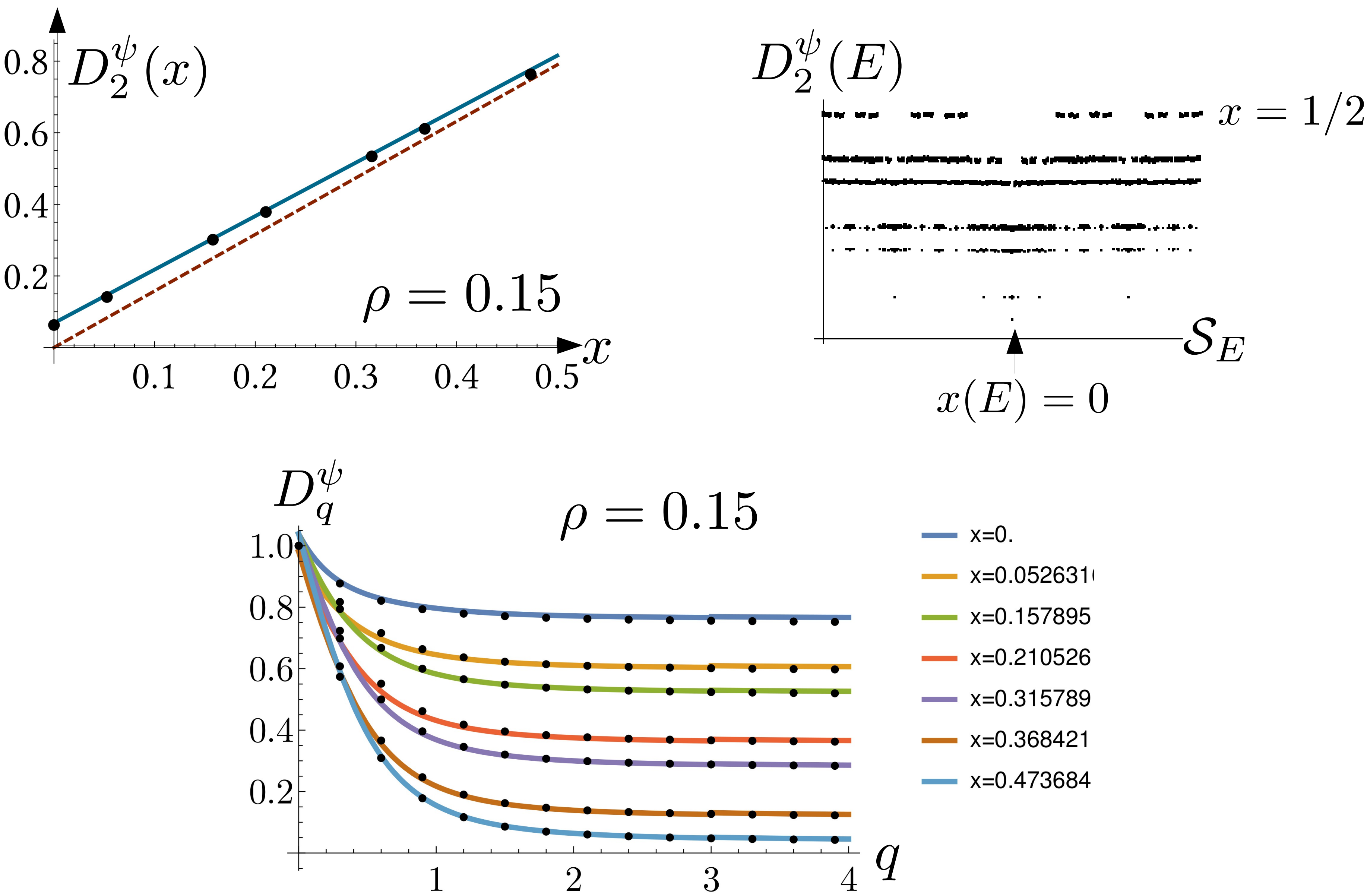
$$\sum_i |\psi_n(i,x(E))|^{2q} = \left(\frac{\bar\lambda^q}{\bar\lambda_q}\right)^{\frac{n(1-2x)}{3}} \left(\frac{\lambda^q}{\lambda_q}\right)^{nx}$$

$$\bar\lambda_q(\rho) = \bar\lambda(\rho^q)$$
$$\lambda_q(\rho) = \lambda(\rho^q)$$

Fractal dimensions :

$$D_q^\psi(x(E)) = \frac{q}{q-1}\left[\frac{(1-2x)}{3}\frac{\log(\bar\lambda/\bar\lambda_q^{1/q})}{\ln\omega} + x\frac{\log(\lambda/\lambda_q^{1/q})}{\ln\omega}\right]$$

Comparison with numerics

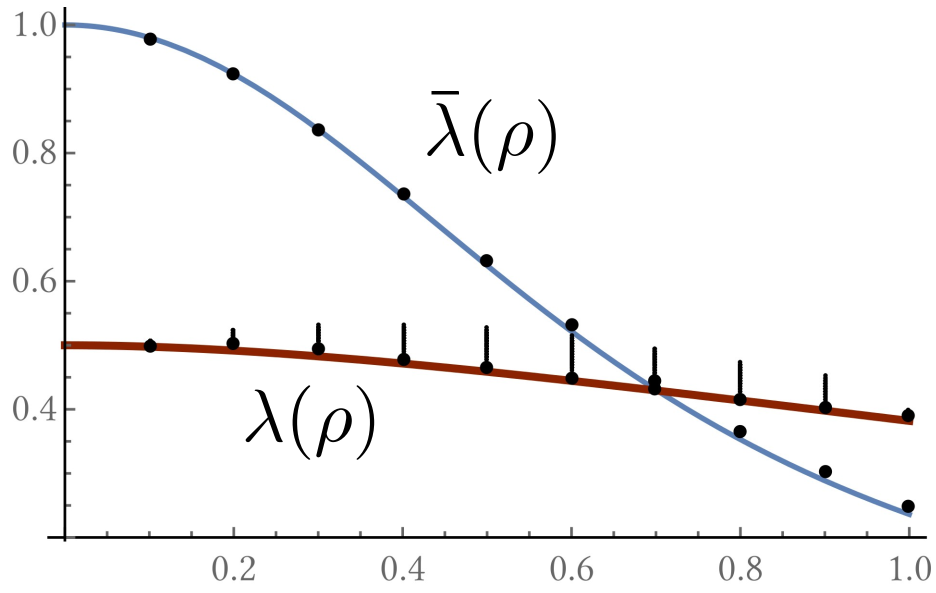


Comparison with numerics

$$D_{\infty}^{\psi}(x) = \frac{(1-2x)}{3}\frac{\log \bar{\lambda}}{\ln \omega} + x\frac{\log \lambda}{\ln \omega}$$

$$D_{\infty}(0) \to \bar{\lambda}(\rho)$$

$$D_{\infty}(1/2) \to \lambda(\rho)$$



Non perturbative expressions: (obtained from sum rules)

$$\bar{\lambda}(\rho) = 2/[(1+\rho^2)^2 + \sqrt{(1+\rho^2)^4 + 4\rho^4}]$$

$$\lambda(\rho) = (1+\rho^2)/[(1+2\rho^2) + \sqrt{(1+\rho^2)^2 + \rho^4}]$$

$$\bar{\lambda}(1) = \omega^3$$

$$\lambda(1) = \omega^2$$

Comparison with numerics

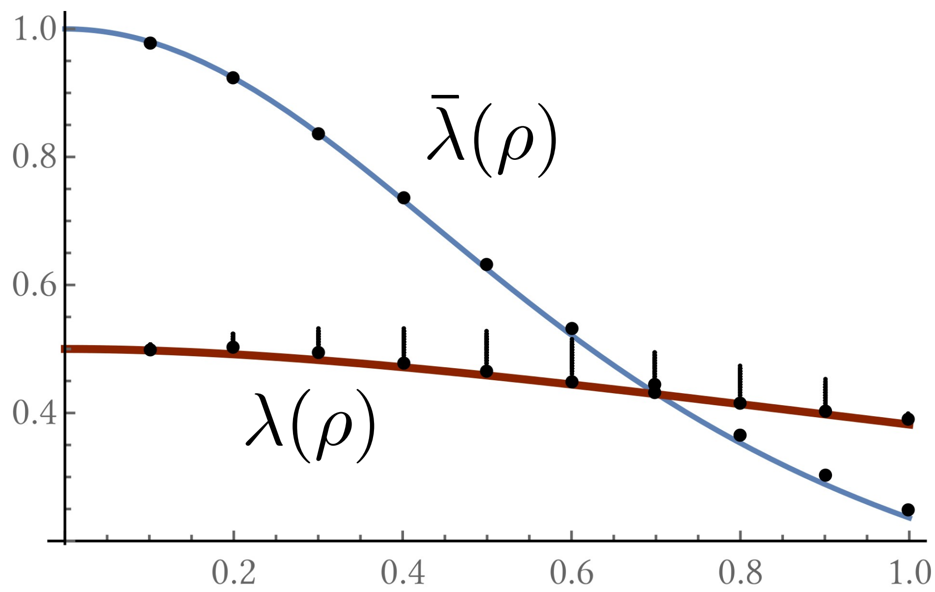
$$D_\infty^\psi(x) = \frac{(1-2x)}{3}\frac{\log\bar{\lambda}}{\ln\omega} + x\frac{\log\lambda}{\ln\omega} \quad \Longleftrightarrow \quad \frac{1}{\alpha(x)} = \frac{1-2x}{3}\frac{\log\bar{\mathrm{z}}}{\log\omega} + x\frac{\log\mathrm{z}}{\log\omega}$$

wave function spatial fractalilty

energy band fractality

$$D_\infty(0) \to \bar{\lambda}(\rho)$$

$$D_\infty(1/2) \to \lambda(\rho)$$



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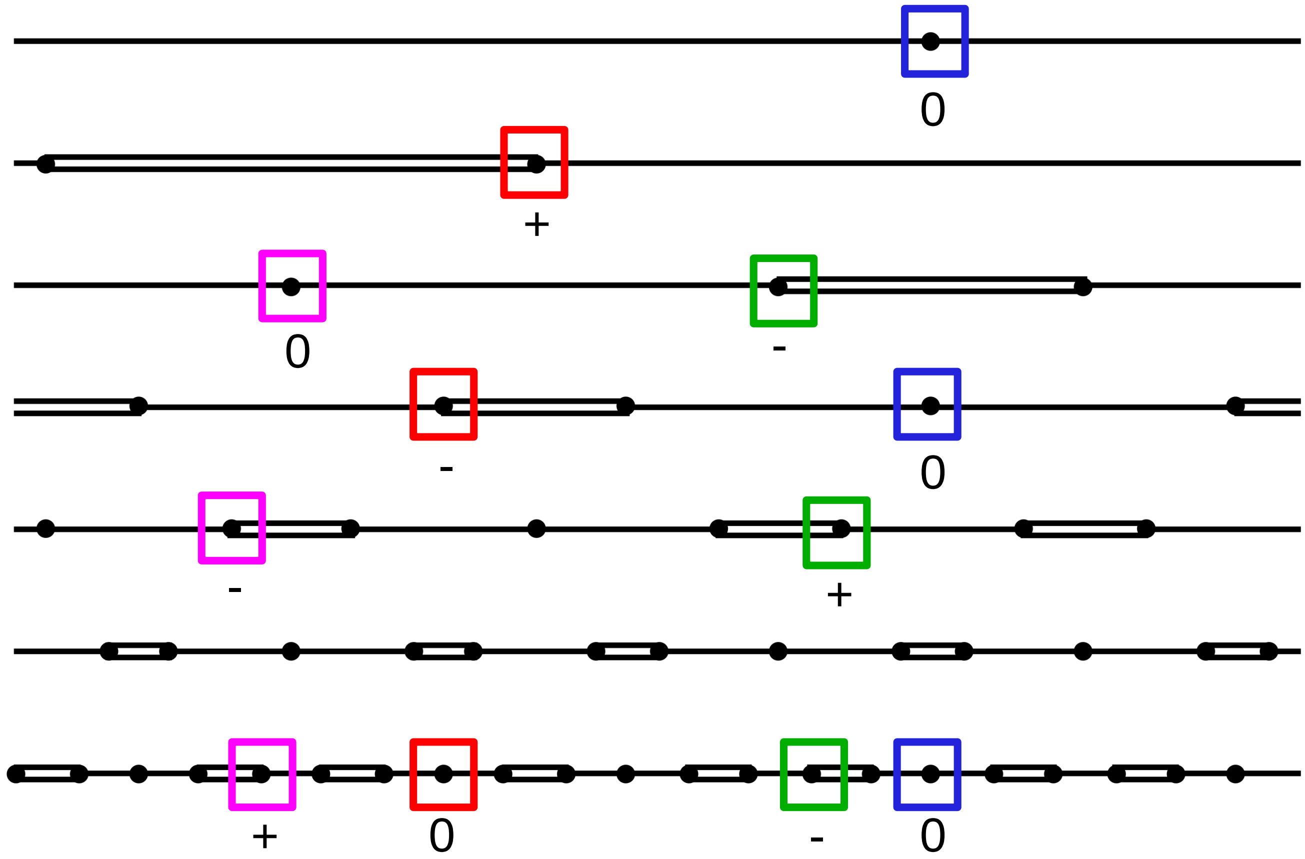
$$\lambda(1) = \omega^2$$

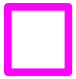
Preferential RG path of sites

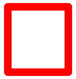
atom → 0
mol. left → -
mol. right → +

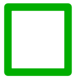
Each site can be labelled by a sequence of three indices :

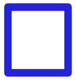
$i \to \mathcal{S}_i = (.. - ..0.. + ..)$



 $= (0 - +)$

 $= (+ - 0)$

 $= (- + -)$

 $= (000)$

$$2(n_+ + n_-) + 3n_0 = n$$

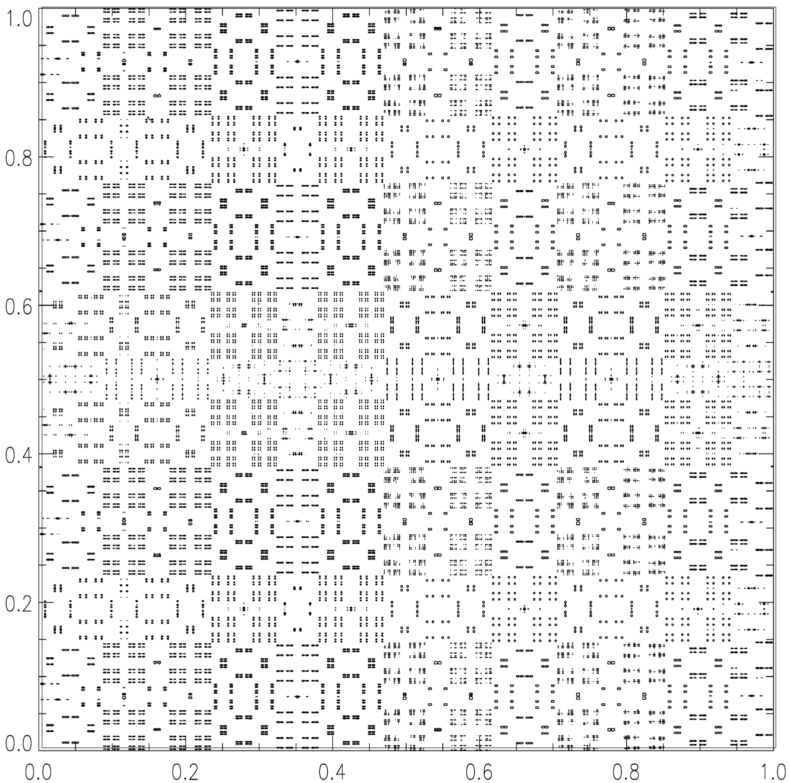
$$x(i) = \frac{n_+ + n_-}{n}$$

Reordering the sites according to $\mathcal{S}_i = (.. - ..0.. + ..)$ is equivalent to conumerotation

Wave functions probability

$$|\psi_n(i, \mathcal{S}_E)|^2$$

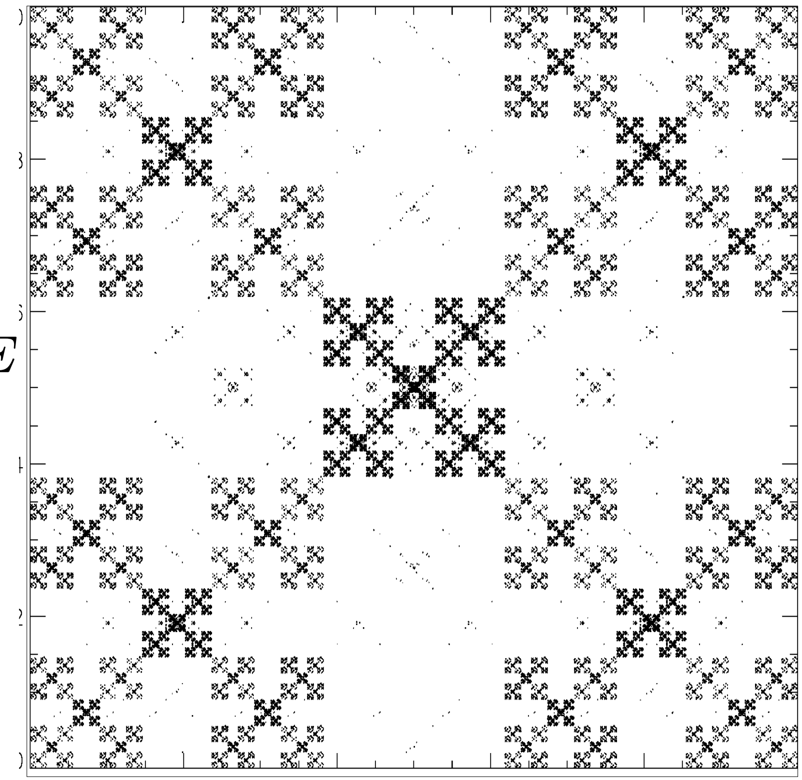
$\mathcal{S}_E$



normal position i

$$|\psi_n(\mathcal{S}_i, \mathcal{S}_E)|^2$$

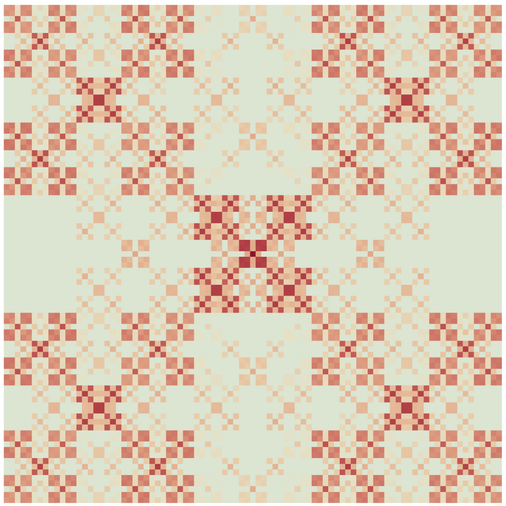
$\mathcal{S}_E$

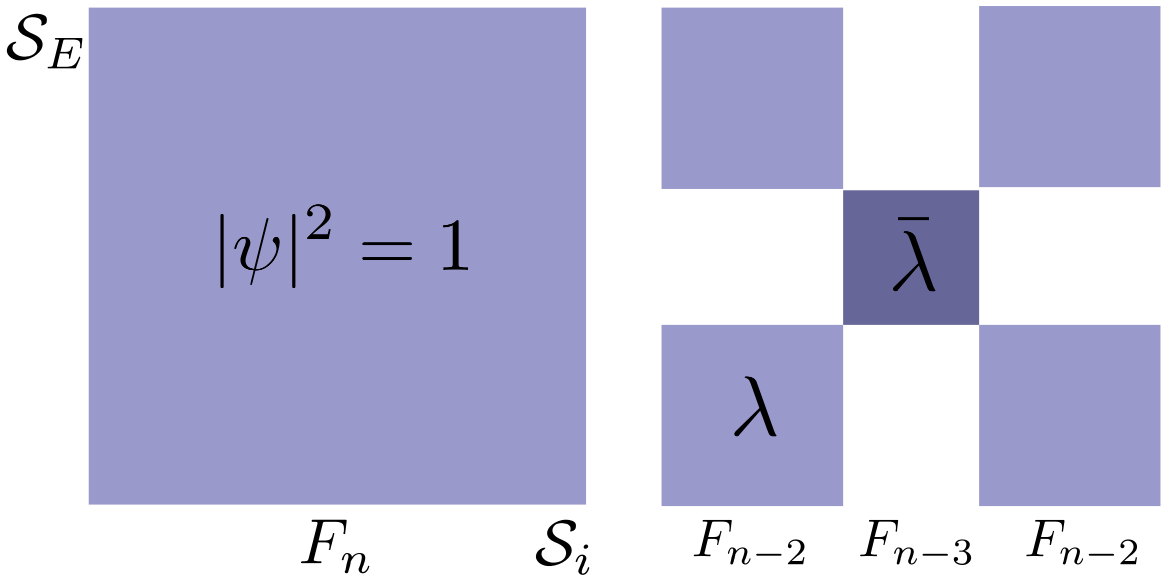


conumeroration position $\mathcal{S}_i$

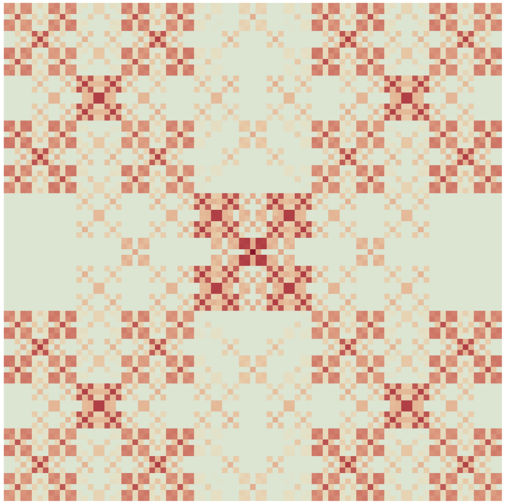
$$\left.\begin{array}{l} \sum_i |\psi_n(i,E)|^{2q} = F_n^{(1-q)D_q^{\psi}(E)} \\ \sum_E |\psi_n(i,E)|^{2q} = F_n^{(1-q)\tilde{D}_q^{\psi}(i)} \end{array}\right\} \longrightarrow D_q^{\psi}(x(E)) = \tilde{D}_q^{\psi}(x(i))$$

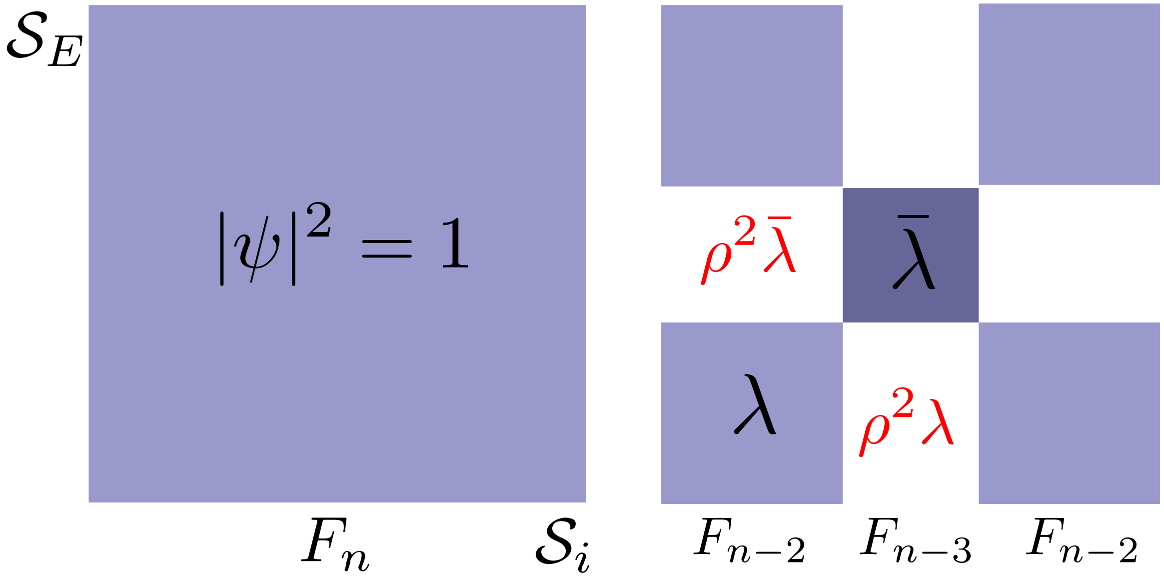
Geometric reconstruction of Wave functions probability



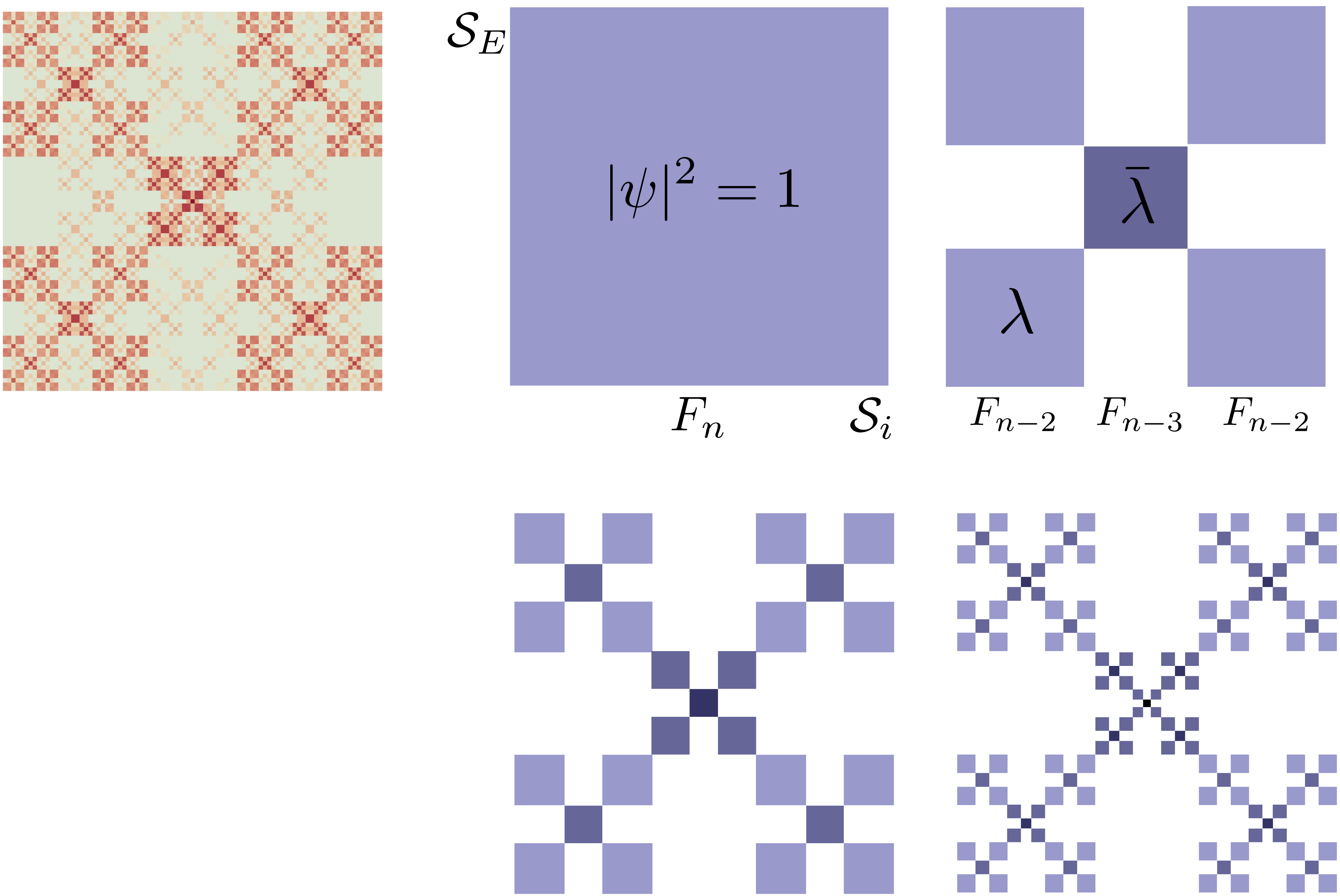


Geometric reconstruction of Wave functions probability





Geometric reconstruction of Wave functions probability

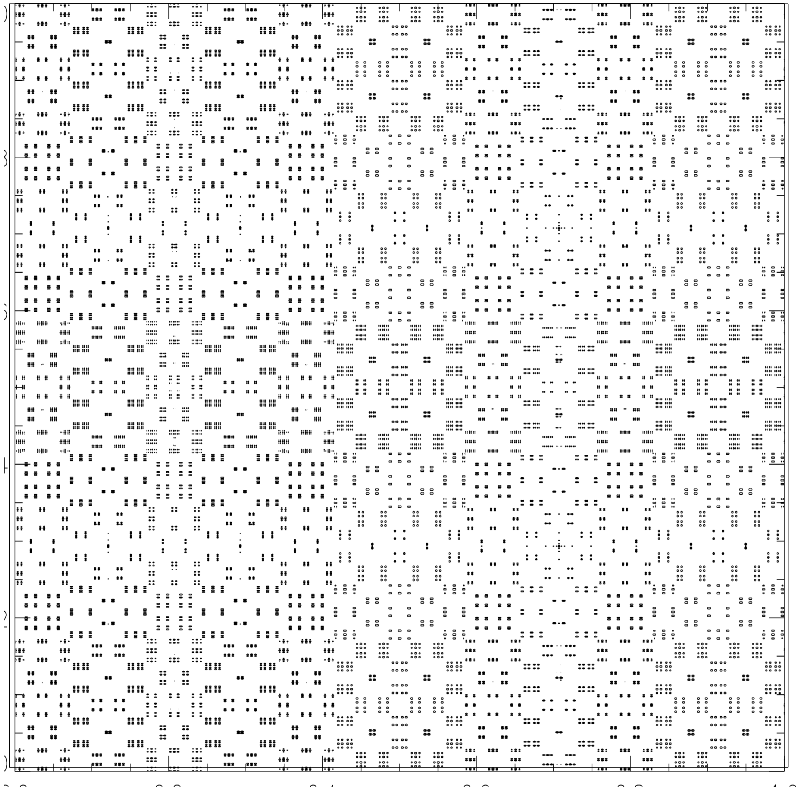


Silver mean quasieriodic chain

$$\omega = 1/(1+\sqrt{2})$$

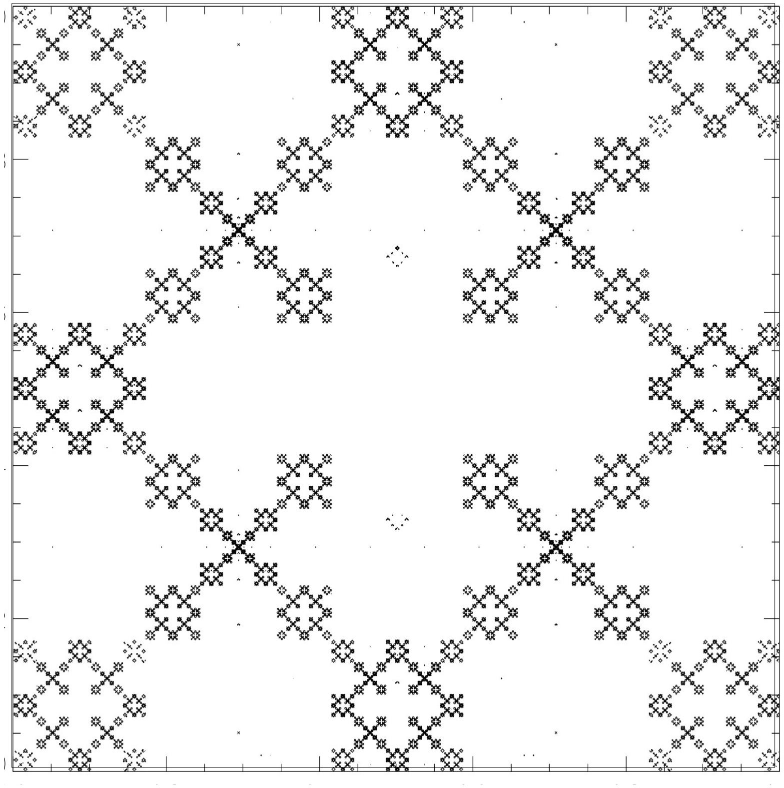
$|\psi_n(i, \mathcal{S}_E)|^2$

$\mathcal{S}_E$



normal position i

$|\psi_n(\mathcal{S}_i, \mathcal{S}_E)|^2$



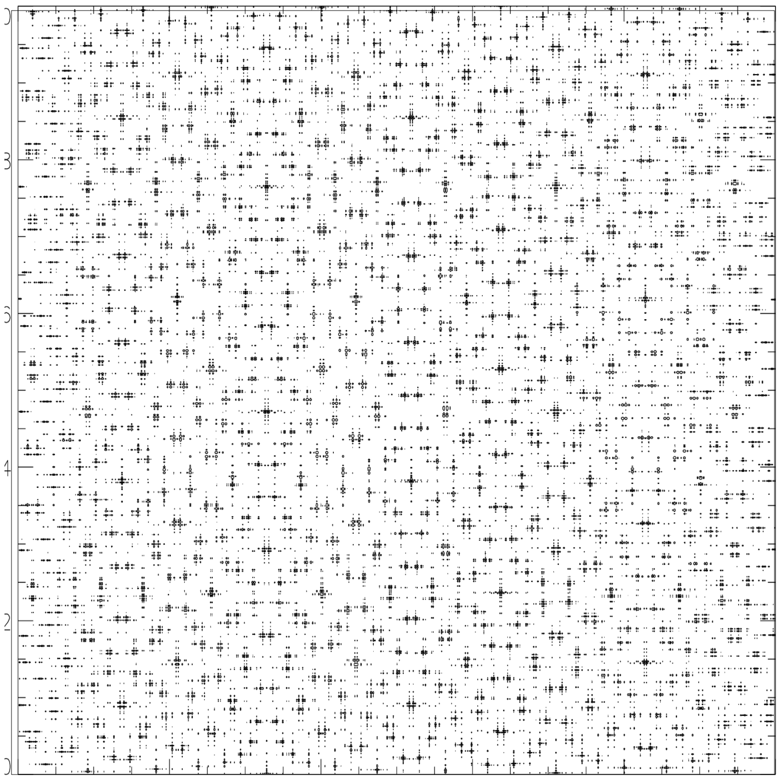
conumeroration position $\mathcal{S}_i$

Harper model

$$E\ \psi_i = t(\psi_{i-1} + \psi_{i+1}) + 2\cos(\omega i + \phi)\psi_i$$

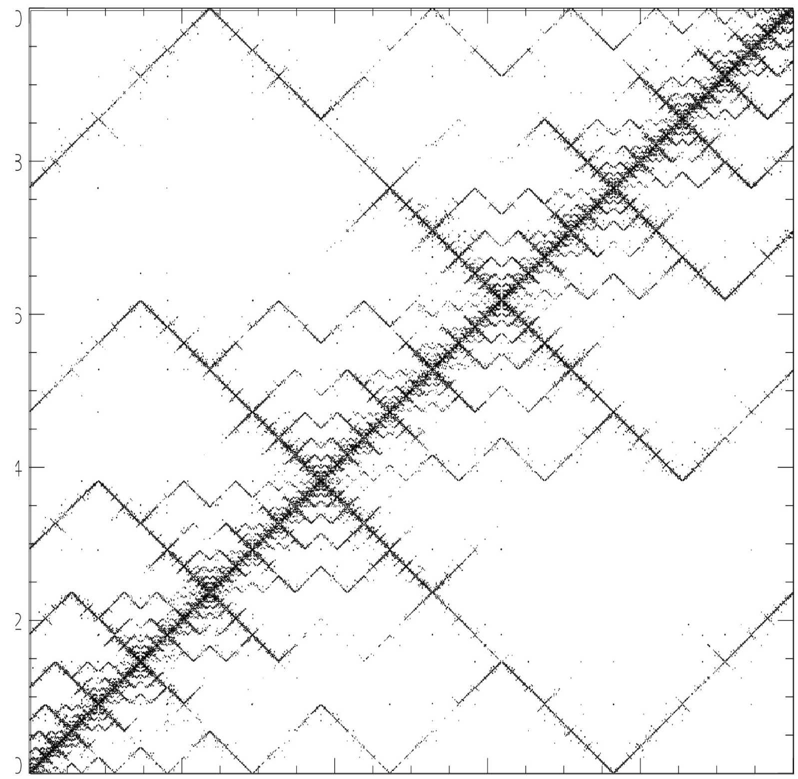
$|\psi_n(i, \mathcal{S}_E)|^2$

$\mathcal{S}_E$



normal position i

$|\psi_n(\mathcal{S}_i, \mathcal{S}_E)|^2$



$$\mathcal{S}_i = \cos(\omega i + \phi)$$

« conumeroration » position

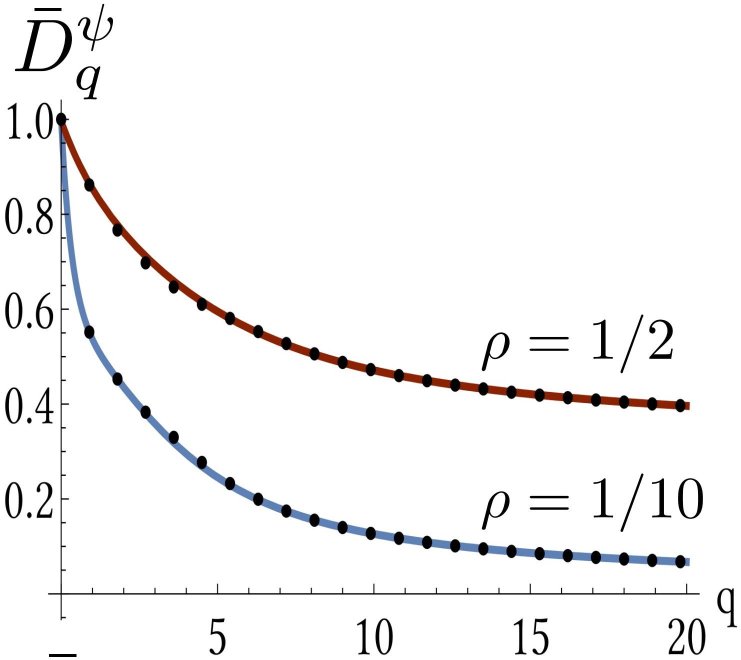
Sites average fractal dimensions

average fractal dimension of wavefunctions:

$$\sum_{i,E} |\psi_n(i,E)|^{2q} = F_n^{(1-q)\bar{D}_q^{\psi}(E)}$$

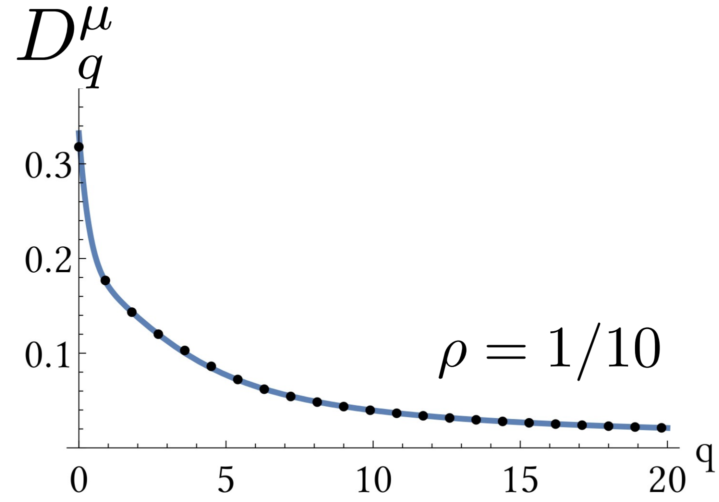
$$2\omega^2 \frac{\lambda^q}{\lambda_q} \omega^{(1-q)\bar{D}_q^{\psi}} + \omega^3 \frac{\bar{\lambda}^q}{\bar{\lambda}_q} \omega^{3(1-q)\bar{D}_q^{\psi}} = 1$$

$$\bar{\lambda}_q(\rho) = \bar{\lambda}(\rho^q)$$
$$\lambda_q(\rho) = \lambda(\rho^q)$$



average fractal dimension of the LDOS:

$$2\omega^2 \frac{\lambda^q}{\lambda_q} z^{(1-q)\bar{D}_q^{\mu}} + \omega^3 \frac{\bar{\lambda}^q}{\bar{\lambda}_q} \bar{z}^{(1-q)\bar{D}_q^{\mu}} = 1$$



Relation between fractal dimensions

average fractal dimension of wave functions:

$$2\omega^2 \frac{\lambda^q}{\lambda_q} \omega^{(1-q)\bar{D}_q^\psi} + \omega^3 \frac{\bar{\lambda}^q}{\bar{\lambda}_q} \omega^{3(1-q)\bar{D}_q^\psi} = 1$$

$$\bar{\lambda}_q(\rho) = \bar{\lambda}(\rho^q)$$

$$\lambda_q(\rho) = \lambda(\rho^q)$$

average fractal dimension of the LDOS:

$$2\omega^2 \frac{\lambda^q}{\lambda_q} \mathrm{z}^{(1-q)\bar{D}_q^\mu} + \omega^3 \frac{\bar{\lambda}^q}{\bar{\lambda}_q} \bar{\mathrm{z}}^{(1-q)\bar{D}_q^\mu} = 1$$

fractal dimension of the DOS:

$$2\omega^{2q} \mathrm{z}^{(1-q)D_q} + \omega^{3q} \bar{\mathrm{z}}^{(1-q)D_q} = 1$$

$$\bar{D}_q^\mu = \bar{D}_q^\psi D_{1+(q-1)\bar{D}_q^\psi}$$

needs for further numerical check !

Summary & perspectives

- Perturbative RG on the Fibonacci chain :
  - -multifractal properties of the energy spectrum
  - -multifractal properties of the wave functions

- For a fixed energy E the scaling exponent of the band $\alpha(E)$ and of the fractal dimensions of wave function $D_q^{\psi}(E)$ depend only on RG path $x(E)$

- Symmetry between energy RG path $\mathcal{S}_E$ and site RG path (conumerotation) $\mathcal{S}_i$ for $x(E) = x(i)$

- Relation between site average fractal dimensions :

$$\bar{D}_q^{\mu} = \bar{D}_q^{\psi} D_{1+(q-1)\bar{D}_q^{\psi}}$$

« LDOS=Wave function x Spectrum »

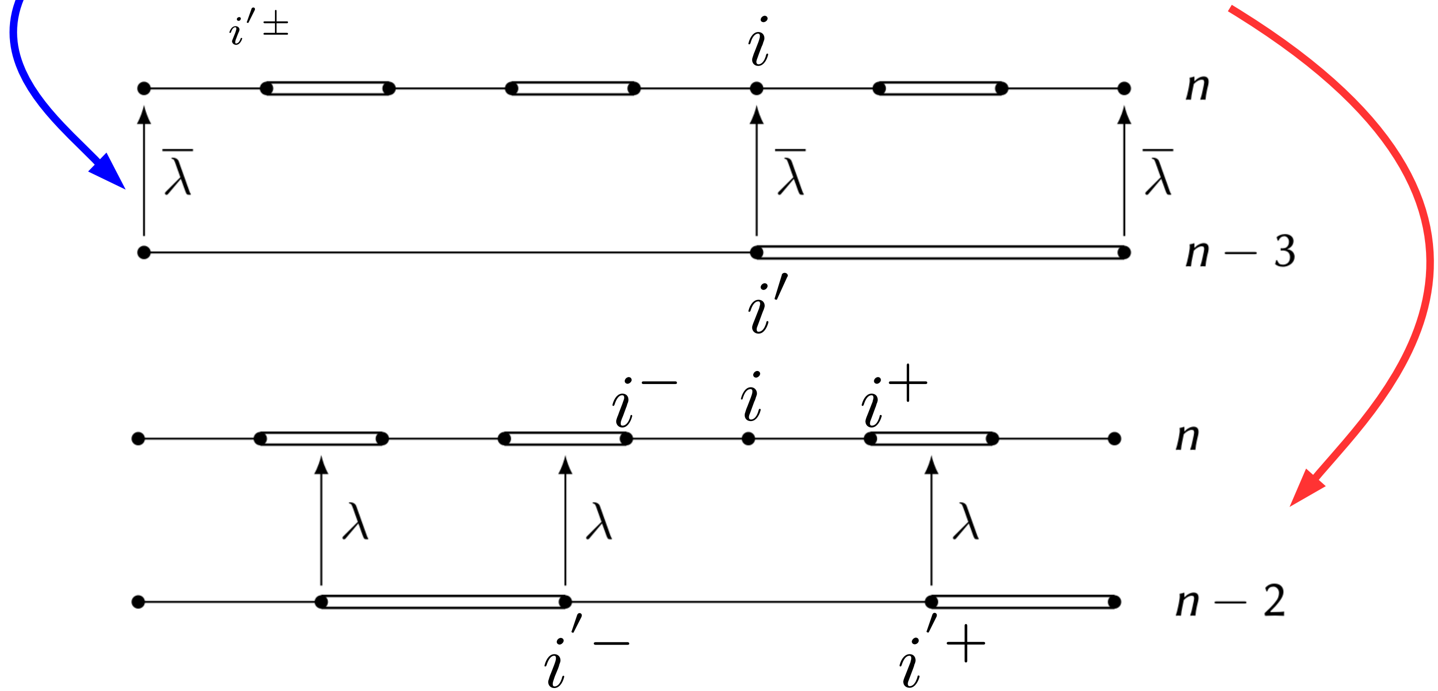
- Work in progress: relation with wave packet anomalous diffusion exponents
- Waves in 2D/3D tilings : multifractality/anomalous diffusion ?
- Phason disorder

Local density of states

atom site :

$$d\mu_n(i,E) \simeq \bar{\lambda}\omega^3 d\mu_{n-3}(i',\bar{\mathrm{z}}E)$$

$$+\sum_{i'^{\pm}} \rho^2\lambda\omega^2[d\mu_{n-2}(i'^{\pm},\mathrm{z}E-t_s)+d\mu_{n-2}(i'^{\pm},\mathrm{z}E+t_s)]$$



molecule site :

$$d\mu_n(i,E) \simeq \lambda\omega^2[d\mu_{n-2}(i',\mathrm{z}E-t_s)+d\mu_{n-2}(i',\mathrm{z}E+t_s)]$$

$$+\rho^2\bar{\lambda}\omega^3 d\mu_{n-3}(i'',\bar{\mathrm{z}}E)$$

Atoms and molecules :

Niu & Nori (1986)
Kalugin,Kitaev,Levitov (1986)

$t_w = 0$

Atoms : isolated sites F_{n-3}

$$E\ \psi_i = t_w(\psi_{i-1} + \psi_{i+1})$$

Molecules : isolated dimer F_{n-2}

$$E\ \psi_i = t_w\ \psi_{i\pm1} + t_s\ \psi_{i\mp1}$$

Energy level	Degeneracy
$E = 0$ (central)	F_{n-3}
$E = \pm t_s$ Bonding/antibonding (lateral)	F_{n-2}

$\rho = t_w / t_S \ll 1$

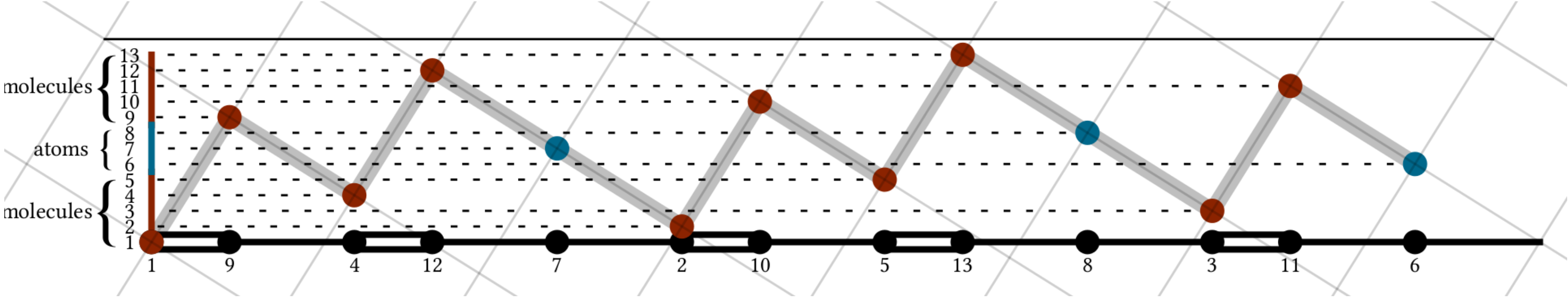
We can build 3 the effective models :

-chain of formed by atom sites : $E = 0$ is splitted into F_{n-3} levels

-chains formed by bonding states : $E = -t_s$ into F_{n-2} levels

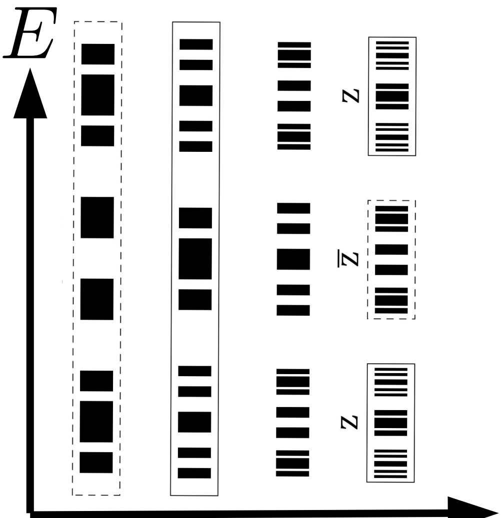
-chains formed by antibonding states : $E = t_s$ into F_{n-2} levels

Conumerotation and RG path of sites



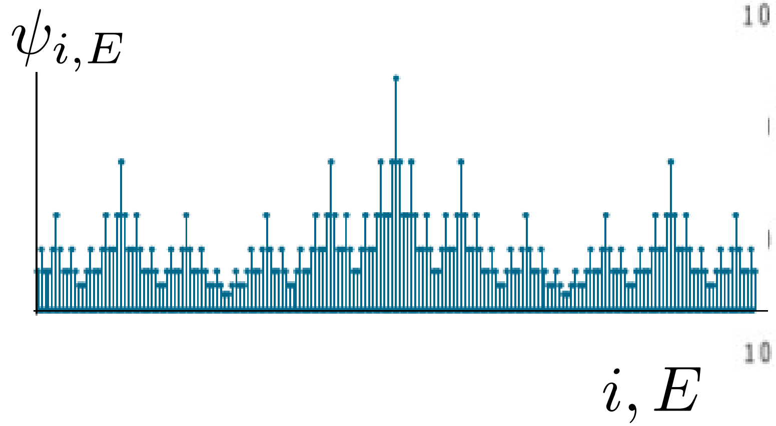
Quasiperiodic chains:

Energy spectrum

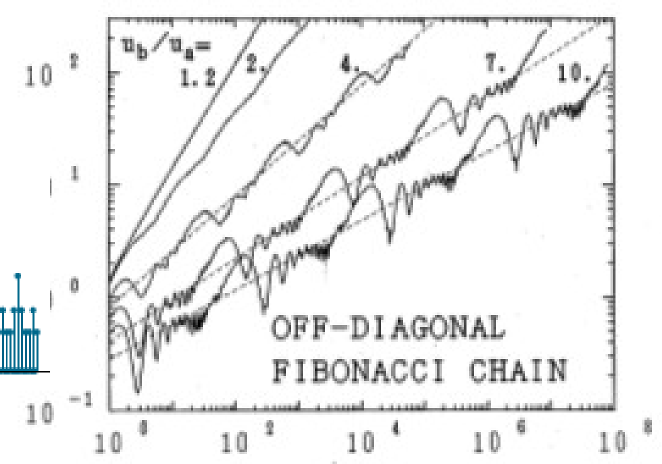


Period of approximant

Wavefunctions



diffusion of wavepacket



$$\langle x^q \rangle \sim t^{q\sigma_q}$$

Fractal dimensions:

Density of states: D_q

Wavefunctions (spatial) : $D_q^{\psi}(E)$

Local Density of states : $D_q^{\mu}(i)$

Diffusion exponents :

$$\sigma_q(i)$$

For a given wavepacket :

$$\sigma_q \geq D_1^{\mu}$$ Guarneri, 1989

$$\sigma_q \geq \frac{D_2^{\mu}}{D_2^{\psi}}$$ Ketzmerick et al, 1997

Initial site average wavepacket :

$$\bar{\sigma}_q = D_{1-q}$$ Piéchon, 1996

Waves in quasiperiodic chains or tiling:

- Quasiperiodic order or geometry induces :

  -multifractal wavefunctions
  -anomalous diffusion of wavepacket

Supplementary difficulty : non trivial energy spectrum (global density of states)
- « topological gaps » : number (infinite/finite) ? Closed/open ? Labelling ?
-multifractal properties ?

Generic and tunable !

- Geometric phason disoder :
  2D/3D : suppress multifractal properties
  leads to delocalization (diffusive regime)

1D : leads to localization ?