Beyond Dirac combs

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Geo-Dyn-2015, November, 30, 2015

1 Introduction

Definition 1.1. An atomic measure $\mu$ is crystalline if:

(a) the support $\Lambda$ of $\mu$ is a locally finite set

(b) the support $S$ of the Fourier transform $\mathcal{F}(\mu)$ of $\mu$ is also a locally finite set $S$.

A subset $S \subset \mathbb{R}^n$ is locally finite if for every $R$, $S \cap B$ is finite when $B = \{|x| \leq R\}$. Equivalently $S$ can be indexed as a sequence of points $\lambda_j$, $j = 0, 1, 2, \ldots$ tending to infinity.

We shall compute genuine Fourier transforms and not diffraction measures.
Dirac combs are crystalline measures. Do there exist other crystalline measures? If $\Lambda$ is a model set the sum $\sum_{\lambda \in \Lambda} \delta_\lambda$ of Dirac masses on $\Lambda$ is not a crystalline measure.

This problem was raised by André-Paul Guinand in 1959 [1]. It was solved recently by Nir Lev and Alexander Olevskii (Section 5 and [3]). They proved the existence of crystalline measures which cannot be reduced to Dirac combs.

Two weeks ago I returned to Guinand’s work and obtained specific examples of crystalline measures. Alexander Olevskii checked this construction.

Such unexpected patterns shall be investigated and might be found elsewhere.
The Fourier transform \( \mathcal{F}(f) = \hat{f} \) of a function \( f \) is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} \exp(-2\pi i x \cdot \xi) f(x) \, dx.
\]

The Dirac mass at \( a \in \mathbb{R}^n \) is denoted by \( \delta_a \) or \( \delta_a(x) \). Let \( \Gamma \subset \mathbb{R}^n \) be a lattice. A Dirac comb is a sum of Dirac masses on \( \Gamma \). The distributional Fourier transform of the Dirac comb \( \mu = \sum_{\gamma \in \Gamma} \delta_{\gamma} \) is the Dirac comb \( \hat{\mu} \) on the dual lattice \( \Gamma^* \). For all testing function \( f \in \mathcal{S}(\mathbb{R}^n) \) we have:
\[
(1) \quad \text{vol}(\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{y \in \Gamma^*} \hat{f}(y)
\]

**Definition 1.2.** Generalized Dirac combs are finite sums \( \mu = \mu_1 + \ldots + \mu_N \) where \( \sigma_j, 1 \leq j \leq N \), is a Dirac comb supported by a coset \( x_j + \Gamma_j \) of a lattice \( \Gamma_j \), \( \mu_j = P_j(x)\sigma_j \), and \( P_j(x) \) is a trigonometric polynomial.

The Fourier transform of a generalized Dirac comb is a generalized Dirac comb.
A.P. Guinand was born in Renmark, South Australia. He went through life with his education paid for by scholarships. Guinand graduated with a B.Sc. with first class honors from the University of Adelaide (1933) and a Ph.D. from Oxford (1937). One of the supervisor’s of his thesis was Hardy.

Guinand is a worldly scholar attending Göttingen, Germany in 1937-38 and Princeton, USA in 1939-40. He served in the RCAF from 1940-45, as a navigator and navigation instructor in Canada until 1943 and then in operational research and experimental navigation in England, eventually making squadron leader. He worked from 1947-1956 at the Royal Military College of Science, Shrivenham, England as a lecturer, then as an associate professor. Guinand was department head at the University of New England, Armidale, N.S.W., Australia, and then the University of Alberta (1957-1960). Guinand was with the U of S from 1960-1964 and was department head from 1962-64. In June 1964 he resigned to take a position at Trent University in Peterborough, Ontario.
Let \( \mu \) be a crystalline measure. We then have \( \mu = \sum_{\lambda \in \Lambda} a(\lambda) \delta_\lambda \) and \( \hat{\mu} = \sum_{y \in S} b(y) \delta_y \) where \((a(\lambda))_{\lambda \in \Lambda}\) and \((b(y))_{y \in S}\) satisfy
\[
(2) \quad a(\lambda) \neq 0, \, \lambda \in \Lambda, \quad b(y) \neq 0, \, y \in S.
\]
and \( \Lambda, S \) are two locally finite sets.

Following Guinand we then say that \( \Lambda \) and \( S \) are in Fourier reciprocity.

Then for all testing function \( f \in \mathcal{S}(\mathbb{R}^n) \) the following generalized Poisson formula holds
\[
(3) \quad \sum_{\lambda \in \Lambda} a(\lambda) \hat{f}(\lambda) = \sum_{y \in S} b(y) f(y)
\]

A trivial example is given by lattices. Does there exist a pair \((\Lambda, S)\) in Fourier reciprocity where \( \Lambda \) is a model set and \( S \) is locally finite? Nobody knows.
Selberg trace formula is similar to (3) but an integral term \( \int f(x) \tanh(\pi x) \, dx \) appears in the right hand side of (3). Yves Colin de Verdières proved a Poisson formula on a Riemannian surface with curvature \(-1\). Then \( \Lambda \) is the collection of lengths of primitive closed geodesics while \( S \) is the normalized spectrum of the Laplace operator. But here also an integral term shows up in the right hand side of (3).
A locally finite set $\Lambda$ is uniformly discrete if

\[
\inf_{\{\lambda, \lambda' \in \Lambda; \lambda \neq \lambda'\}} |\lambda - \lambda'| = \beta > 0
\]

Nir Lev and Alexander Olevskii [2] proved the following

**Theorem 1.1.** In one dimension if both the support $\Lambda$ of an atomic measure $\mu$ and the support $S$ of its Fourier transform are uniformly discrete sets, then $\mu$ is a generalized Dirac comb.

The problem is still open in dimensions $n \geq 2$.

If $\Lambda$ and $S$ are two locally finite sets but not uniformly discrete this is not the case. Indeed Lev and Olevskii proved the following in any dimension [3]

**Theorem 1.2.** There exists a measure supported by a locally finite set whose Fourier transform is also supported by a locally finite set and which is not a generalized Dirac comb.
2 Guinand’s approach

We prove Theorem 1.2 in one dimension. Legendre theorem says that an integer \( n \geq 0 \) can be written as a sum of three squares if and only if \( n \) is not of the form \( 4^j(8k + 7) \). For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares (1=1+0+0, etc). But 7 is not.

Let \( r_3(n) \) be the number of decompositions of the integer \( n \geq 1 \) into a sum of three squares (with \( r_3(n) = 0 \) if \( n \) is not a sum of three squares). More precisely \( r_3(n) \) is the number of points \( k \in \mathbb{Z}^3 \) such that \( |k|^2 = n \). We have \( r_3(4n) = r_3(n) \), \( r_3(0) = 1 \), \( r_3(1) = 6 \), . . .

Guinand began his seminal work [1] with a lemma

**Lemma 2.1.** For all \( x > 0 \) we have

\[
1 + \sum_{1}^{\infty} r_3(n) \exp(-\pi nx) = x^{-3/2} +
\]

\[
(5) \quad x^{-3/2} \sum_{1}^{\infty} r_3(n) \exp(-\pi n/x)
\]
Indeed the standard Poisson formula yields

\[ (6) \quad \sum_{\infty}^{\infty} \exp(-\pi k^2 x) = x^{-1/2} \sum_{\infty}^{\infty} \exp(-\pi k^2 / x) \]

and it suffices to raise this identity to the cubic power.

Guinand then viewed \( x > 0 \) as a parameter and considered the family of odd functions \( f_x(t) = t \exp(-\pi xt^2) \) of the argument \( t \). Then (5) can be written

\[
\frac{df_x}{dt}(0) + \sum_{1}^{\infty} r_3(n)n^{-1/2} f_x(\sqrt{n}) =
\]

\[ i \frac{d\hat{f}_x}{dt}(0) + i \sum_{1}^{\infty} r_3(n)n^{-1/2} \hat{f}_x(\sqrt{n}) \]

Guinand introduced the odd distribution

\[ (8) \quad \sigma = -2 \frac{d}{dt} \delta_0 + \sum_{1}^{\infty} r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) \]

We have \( \sum_{0}^{N} r_3(n)n^{-1/2} \sim 2\pi N \) which implies that \( \sigma \) is a tempered distribution.
Guinand proved the following

**Lemma 2.2.** The distributional Fourier transform of $\sigma$ is $-i\sigma$.

Indeed $f_x(t) = t \exp(-\pi xt^2)$ and (7) can be rewritten as $\langle \sigma, f_x \rangle = i\langle \sigma, \hat{f}_x \rangle$ or $\langle \sigma, f_x \rangle = i\langle \hat{\sigma}, f_x \rangle$. But the collection of odd functions $f_x$, $x > 0$, is total in the subspace of odd functions of the Schwartz class. This implies Lemma 2.2.

A variant on Guinand’s distribution $\sigma$ is the measure $\tilde{\sigma} = -4\pi t + \sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$. Since $\mathcal{F}(\frac{d}{dt}\delta_0 - 2\pi t) = -i(\frac{d}{dt}\delta_0 - 2\pi t)$ we also have

$$\mathcal{F}(\tilde{\sigma}) = -i\tilde{\sigma}.$$ 

We now move one small step beyond Guinand’s work and prove Theorem 1.2. Let $\alpha \in (0, 1)$ and set

$$\tau_\alpha(t) = (\alpha^2 + \frac{1}{\alpha}) \sigma(t) - \alpha \sigma(\alpha t) - \sigma(t/\alpha)$$

Then $\frac{d}{dt}\delta_0$ disappears in this linear combination. On the Fourier transform side

$$\hat{\tau}_\alpha(y) = (\alpha^2 + \frac{1}{\alpha}) \hat{\sigma}(y) - \hat{\sigma}(y/\alpha) - \alpha \hat{\sigma}(\alpha y) = -i\tau_\alpha$$
We had \( \sigma([1, N]) \approx N^2 \) as \( N \to \infty \) while \( \tau_\alpha([1, N]) \approx N \) which is due to the extra cancellation brought by the linear combination in (9).

If \( \alpha = 1/2 \) in the preceding construction we define \( \chi(n) = -1/2 \) if \( n \geq 1, n \notin 4\mathbb{N} \), \( \chi(n) = 7/4 \) if \( n \in 4\mathbb{N}, n \notin 16\mathbb{N} \) and \( \chi(n) = 3/4 \) if \( n \in 16\mathbb{N} \). The mean value of \( \chi \) is 0. Then we have

**Theorem 2.1.** The Fourier transform of the measure

\[
\tau_{1/2} = \sum_1^\infty \chi(n)r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})
\]

is \(-i\tau_{1/2}\).

Notice that

\[
\sum_1^\infty r_3(n)n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) = \sum_{k \in \mathbb{Z}^3, k \neq 0} \frac{1}{|k|}(\delta_{|k|} - \delta_{-|k|})
\]

which paves the road to our third example (next section).
We turn to our second example. Observe that for every function $f$ the Fourier transform of $\cos(\pi x)[f(x - 1/2) - f(x + 1/2)]$ is $i \cos(\pi \xi)[\hat{f}(\xi - 1/2) - \hat{f}(\xi + 1/2)]$. This simple observation leads to a variant on the measure $\tau$ of Theorem 2.1. Let $\sigma$ be the Guinand distribution and consider the measure $\rho = \cos(\pi x)[\sigma(x - 1/2) - \sigma(x + 1/2)]$. The derivative of the Dirac mass $\frac{d\delta_0}{dx}$ is moved at $1/2$ and $-1/2$ and then transformed into Dirac masses after being multiplied by $\cos(\pi x)$. On the Fourier transform side the derivative of the Dirac mass $\frac{d\delta_0}{dy}$ is transformed into a Dirac mass after multiplication by $\sin(\pi y)$ and then the resulting measure is translated by $\pm 1/2$. Finally the Fourier transform of $\rho$ is $\rho$. We have $\rho =$

$$2\pi \delta_{1/2} + 2\pi \delta_{-1/2} + \sum_{1}^{\infty} \sin(\pi \sqrt{n}) r_3(n) n^{-1/2} (\delta(\sqrt{n} + 1/2) + \delta(\sqrt{n} - 1/2) + \delta(-\sqrt{n} + 1/2) + \delta(-\sqrt{n} - 1/2))$$
3 The third example

Let us begin with a one dimensional example.

**Theorem 3.1.** Let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \not\in \mathbb{Z}^3 \) and \( \beta = (\beta_1, \beta_2, \beta_3) \not\in \mathbb{Z}^3 \). Then the distributional Fourier transform of the measure

\[
\sigma_{(\alpha,\beta)} = \sum_{k \in \mathbb{Z}^3} \frac{\exp(2\pi ik \cdot \beta)}{|k + \alpha|} (\delta_{|k+\alpha|} - \delta_{-|k+\alpha|})
\]

is

\[
\mathcal{F}(\sigma_{(\alpha,\beta)}) = -i \exp(-2\pi i \alpha \cdot \beta) \overline{\sigma_{(\beta,\alpha)}}
\]

The Fourier transform of the measure

\[
\sum_{k \in \mathbb{Z}^3} \frac{1}{|k + \beta|} (\delta_{|k+\beta|} - \delta_{-|k+\beta|})
\]

is not a measure. Cancellations are introduced in \( \sigma_{(\alpha,\beta)} \) by the phase factor \( \exp(2\pi ik \cdot \alpha) \) and play a seminal role.

If \( 1, \beta_1, \beta_2, \beta_3 \) are linearly independant over \( \mathbb{Q} \) then for \( k \neq l, k, l \in \mathbb{Z}^3, |k + \beta| \neq |l + \beta| \). It implies \( \int_x^{x+1} d|\sigma_{(\alpha,\beta)}|(t) \simeq x, x \to \infty \), and this estimate is optimal.
Therefore \(\sigma_{(\alpha,\beta)}\) is a tempered measure which is not an almost periodic measure. Our second construction yields an almost periodic measure.

We have \(\sigma_{(\alpha,\beta)} = \sigma_{(-\alpha,-\beta)}\). If \(\alpha_1 = \beta_1 = 1/2, \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = 0\), then \(\sigma_{(\alpha,\beta)} = 0\).

There exist infinitely many \(\alpha \in \mathbb{R}^3\) such that the set of all \(|k + \alpha|, k \in \mathbb{Z}^3\), is linearly independent over \(\mathbb{Q}\).

It is interesting to let \(\alpha\) and \(\beta\) tend to 0. The limit of \(\sigma_{(\alpha,\beta)}\) is the Guinand distribution

\[
-2 \frac{d}{dt} \delta_0 + \sum_{k \in \mathbb{Z}^3; k \neq 0} \frac{1}{|k|} (\delta_{|k|} - \delta_{-|k|}).
\]

Theorem 3.1 implies that the Fourier transform of the Guinand distribution \(\sigma_0\) is \(-i\sigma_0\).

Here is a two dimensional example. Choose \(A \in SO(4, \mathbb{R})\) and let \(A'\) be the \(3 \times 4\) matrix obtained by deleting the last row \(a\) from \(A\). Let \(\alpha = (\alpha_1, \alpha_2, \alpha_3) \notin \mathbb{Z}^3\).
Theorem 3.2. The distributional Fourier transform of the (two dimensional) measure

\[ \sigma_\alpha = \sum_{k \in \mathbb{Z}^4} \frac{\exp(2\pi i A'(k) \cdot \alpha)}{|A'(k) + \alpha|} \left( \delta(|A'(k) + \alpha|, a \cdot k) - \delta(-|A'(k) + \alpha|, a \cdot k) \right) \]

is

\[ \mathcal{F}(\sigma_\alpha) = -\frac{i}{2} \exp(-2\pi i |\alpha|^2) \overline{\sigma_\alpha} \]

Here also the Fourier transform of

\[ \sum_{k \in \mathbb{Z}^4} \frac{1}{|A'(k) + \alpha|} \left( \delta(|A'(k) + \alpha|, a \cdot k) - \delta(-|A'(k) + \alpha|, a \cdot k) \right) \]

is not a measure and the phase factors are playing a seminal role.
Proof of Theorem 3.1.

The measure $\sigma_{(\alpha,\beta)}$ is odd. If $\tau$ is an odd measure and if we need to check the identity
\[
\langle \hat{\phi}, \sigma_{(\alpha,\beta)} \rangle = \langle \phi, \tau \rangle
\]
for every testing function $\phi$ it suffices to do it for every odd $\phi$. Let us write $\varphi = \hat{\phi}$. Then the left hand side of (11) is
\[
s(\varphi) = 2 \sum_{k \in \mathbb{Z}^3} \frac{\exp(2\pi ik \cdot \beta)}{|k + \alpha|} \varphi(|k + \alpha|)
\]
We introduce the radial function $\Phi(x) = \frac{\varphi(|x|)}{|x|}$. Then
\[
s(\varphi) = 2 \sum_{k \in \mathbb{Z}^3} \exp(2\pi ik \cdot \beta) \Phi(k + \alpha)
\]
We then use the ordinary Poisson formula to compute $s(\varphi)$.

First step. the Fourier transform of a radial function $f \in L^1(\mathbb{R}^3)$ is
\[
\hat{f}(y) = 4\pi \int_0^{\infty} f(r) \frac{\sin(2\pi |y|r)}{2\pi |y|r} r^2 \, dr
\]
Therefore
\[
\mathcal{F}\left(\frac{\varphi(|\cdot|)}{|\cdot|}\right) = i\frac{\phi(|\cdot|)}{|\cdot|}
\]
or
\[
\mathcal{F}\left(\frac{\phi(|\cdot|)}{|\cdot|}\right) = -i\frac{\varphi(|\cdot|)}{|\cdot|}
\]
with \(\varphi = \hat{\phi}\). A three dimensional Fourier transform is reduced to a one dimensional one.

**Second step.** Poisson formula yields
\[
\sum_{k \in \mathbb{Z}^3} \exp(2\pi ik \cdot \beta) \hat{f}(k + \alpha) = 
\]

\[
\exp(-2\pi i \beta \cdot \alpha) \sum_{k \in \mathbb{Z}^3} \exp(-2\pi ik \cdot \alpha) f(k + \beta)
\]
for every testing function \(f\).

**Third step.** Combining (16) and (17) yields
\[
s(\varphi) = 
\]

\[
i \exp(-2\pi i \alpha \cdot \beta) \sum_{k \in \mathbb{Z}^3} \exp(-2\pi ik \cdot \alpha) |k + \beta|^{-1} \phi(|k + \beta|)
\]
This is \(\langle \tau, \phi \rangle\) which ends the proof.
4 Fourth approach

The fourth proof works in any dimension. Fix the real number $\alpha$ in $(0, 1/4)$. The measure $\sigma$ will be constructed as a series

\[(18) \quad \sigma = \sum_{0}^{\infty} \epsilon_j \sigma_j\]

where

\[(19) \quad \sigma_j \text{ is a generalized Dirac comb and is not the zero measure}\]

\[(20) \quad \text{the support of } \sigma_j \text{ is contained in } \Lambda_j = 2^{-j}(\mathbb{Z} + 1/2) \cap \{|x - 1/2| > \alpha 2^j\}\]

More precisely this support is the union of the intervals

\[|x - 1/2 - (k + 1/2)2^j| \leq (1/2 - \alpha)2^j, \ k \in \mathbb{Z}\]

\[(21) \quad \sigma_j \text{ is } 2^j-\text{periodic}\]

\[(22) \quad \text{the support of the Fourier transform } \hat{\sigma}_j \text{ of } \sigma_j \text{ is also contained in } \Lambda_j.\]

\[(23) \quad \text{the choice of } \epsilon_j > 0 \text{ ensures the convergence of the series } \sigma = \sum_{0}^{\infty} \epsilon_j \|\sigma_j\|_M.\]
Observe that $\Lambda_j$ are pairwise disjoint sets in such a way that $\sigma$ is not a generalized Dirac comb. Moreover $\bigcup_0^\infty \Lambda_j$ is a locally finite set. Theorem 2.1 will be proved once these $\sigma_j$ will be constructed.

How does one construct these $\sigma_j$? Here is the recipe. If $(\Lambda, S)$ is in Fourier reciprocity, so are $(\Lambda + u, S + \tau)$ for any $u, \tau \in \mathbb{R}^n$. This implies that one can reduce the construction of $\sigma_j$ to the case where

$$\Lambda_j = \{x \in 2^{-j}\mathbb{Z}; \ |x| > \alpha 2^j\}$$

It then suffices to use the following lemma:

**Lemma 4.1.** Let $N \geq 2$ be an integer and $\alpha \in (0, 1/4)$. There exists an $N$–periodic measure which is a sum $\sigma$ of Dirac masses on $N^{-1}\mathbb{Z} \cap \{x; |x| \geq \alpha N\}$ and whose Fourier transform is also supported by $\Lambda_N$. Moreover the support of $\sigma$ is $\Lambda_N = \bigcup_{k \in \mathbb{Z}} I_k$ where $I_k = [(k + 1/2)N - (1/2 - \alpha)N, (k + 1/2)N + (1/2 - \alpha)N]$.

Let us prove this lemma. Since $\sigma$ is $N$–periodic we have

$$\sigma = \tau \ast \nu$$
where $\nu$ is the Dirac comb $\sum_{k \in \mathbb{Z}} \delta_{kN}$ and
\[
\tau = \sum_{0}^{N^2-1} c_k \delta_{k/N}.
\]
It implies
\[
\hat{\sigma}(\xi) = N^{-1} \sum_{m \in \mathbb{Z}} P(m) \delta_{m/N}
\]
with $P(\xi) = \sum_{0}^{N^2-1} c_k \exp(-2\pi i k \xi N^{-2})$.

Finally $N$–periodic measures on $N^{-1}\mathbb{Z}$ are in a $1–1$ correspondance with trigonometric polynomials of degree less than $N^2$.

**Lemma 4.2.** Let $M \in \mathbb{N}$, $E, F \subset \mathbb{Z}/M\mathbb{Z}$ be two sets of cardinality $|E|, |F|$.

If $|E| + |F| < M$ there exists a non trivial trigonometric polynomial
\[
P(\xi) = \sum_{0}^{M-1} c_k \exp(2\pi i k \xi / M)
\]
such that
\[
c_k = 0, k \in E, \quad P(\xi) = 0, \xi \in F
\]
Moreover if $E$ and $F$ are two intervals and $|E| + |F| < M/2$ we can impose $c_k \neq 0$, $k \notin E$. 
A simple dimension counting argument implies the first statement.

The proof of the second runs by contradiction. Assuming that the linear space $G$ defined by (17) is not contained in $c_k \neq 0$, $k \notin E$, it implies that $G$ is contained in $c_{k_0} = 0$ for some $k_0 \notin E$. By linear algebra this implies the existence of coefficients $a_k$, $k \in E$, $b_l$, $l \in F$, such that for all trigonometric polynomial

$$P(\xi) = \sum c_k \exp(2\pi ik\xi/M)$$

we have

(28) $$c_{k_0} = \sum_{k \in E} a_k c_k + \sum_{l \in F} b_l P(l)$$

It implies

(29) $$\sum_{l \in F} b_l \exp(2\pi ik_0 l/M) = 1$$

(30) $$\sum_{l \in F} b_l \exp(2\pi ik l/M) = 0, \ k \notin E, k \neq k_0$$

The set $\mathbb{Z}/M\mathbb{Z} \setminus E \cup \{k_0\}$ contains an interval $J$ of length equal to $|F|$ and the matrix

$$((\exp(2\pi ik l/M)))_{k \in J, l \in F}$$

is an inversible Vandermonde matrix. Therefore (30) implies $b_l = 0$, $l \in F$, which contradicts (28).
We now conclude the proof of Lemma 3.1. Our first demand is that \( \sigma \) be an \( N \)-periodic measure carried by \( N^{-1}\mathbb{Z} \setminus N\mathbb{Z} + [-\alpha N, \alpha N] \). The restriction of this measure \( \sigma \) to \([0, N)\) is \( \tau_N = \sum_{k \in T_N} c(k, N) \delta_{kN^{-1}} \) where \[
abla T_N = \mathbb{Z} \cap (\alpha N^2, (1 - \alpha)N^2).\]

Then (15) yields

\[
\hat{\sigma} = N^{-1} \sum_{-\infty}^{\infty} P(l) \delta_{lN^{-1}}
\]

where

\[
P(x) = \hat{\tau}_N(x) = \sum_{k \in T_N} c(k, N) \exp(-2\pi ikN^{-2}x)
\]

We have \((1 - 2\alpha)N^2 - 1 \leq |T_N| \leq (1 - 2\alpha)N^2 + 1\). Lemma 3.1 with \( E_N = [0, N^2 - 1] \setminus T_N \) and \( F_N = ([0, \alpha N^2] \cup [(1 - \alpha)N^2, N^2]) \cap \mathbb{Z} \) yields non trivial coefficients \( c(k, j) \) such that \( P_j(l) = 0 \) when \( |l| \leq \alpha N^2 \) which ends the proof. It works if \( |E_j| + |F_j| < N^2 \) which reads \( \alpha < 1/4 \).
5 The fifth proof (Lev and Olevskii)

Here is the original proof by Lev and Olevskii. The reader will see that our fourth proof mimics this one. Let \( \Gamma \subset \mathbb{R}^n \times \mathbb{R} \) be an oblique lattice. It means that the two projections \( p_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \), \( p_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \), once restricted to \( \Gamma \), are injective with a dense range. The model set \( \Lambda_I \) is defined by the standard cut and projection scheme. The window is the interval \( I = [-a, a] \). Then

\[
\Lambda_I = \{ \lambda = p_1(\gamma); \gamma \in \Gamma, p_2(\gamma) \in I \}
\]

Let \( 0 < h_1 < h_2 < \ldots \) be an increasing sequence of positive numbers tending to infinity. We set \( I_1 = [-h_1, h_1] \subset I_2 = [-h_2, h_2] \subset \ldots \) and we have \( \bigcup_1^\infty I_k = \mathbb{R} \). The corresponding sequence of model sets is \( \Lambda_k, k \in \mathbb{N} \). We have

\[
\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_k \subset \ldots
\]

and \( \bigcup_1^\infty \Lambda_k = p_1(\Gamma) \) is dense in \( \mathbb{R}^n \).

Now we can define projective model sets.
Definition 5.1. Let $a_0 = 0 < a_1 < a_2 < \ldots$ be an increasing sequence of positive numbers tending to infinity. A **projective model set** is defined by

\begin{equation}
\tilde{\Lambda} = \bigcup_{1}^{\infty} \tilde{\Lambda}_k
\end{equation}

where

\begin{equation}
\tilde{\Lambda}_k = \{ \lambda \in \Lambda_k; |\lambda| \geq a_{k-1} \}
\end{equation}

Lemma 5.1. Let $E = \{ k + m\sqrt{2}; (k, m) \in \mathbb{N}^2 \}$. Then $\Lambda = E \cup (-E)$ is a projective model set.

For proving this lemma it suffices to use Definition 5.1 with

$\Gamma = \{(k + m\sqrt{2}, k - m\sqrt{2}); k, m \in \mathbb{Z} \}, h_k = a_k = k$.

A **projective model set** is locally finite. A model set is never a **projective model set**. The density of a model set is finite while the density of a **projective model set** is infinite.
Theorem 5.1. Every projective model set $\tilde{\Lambda}$ contains the support of a measure $\mu$ such that

(a) $\mu$ is not a generalized Dirac comb

(b) the Fourier transform $\hat{\mu}$ of $\mu$ is also supported by a projective model set $S$.

The proof relies on projective model sets which will be defined below. Let $\Gamma \subset \mathbb{R}^n \times \mathbb{R}$ be an oblique lattice. It means that the two projections $p_1 : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$, $p_2 : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$, once restricted to $\Gamma$, are injective with a dense range. The model set $\Lambda_I$ is defined by the standard cut and projection scheme. The window is the interval $I = [-a, a]$. Then, as in [2], [3], [4], [5],

$$\Lambda_I = \{ \lambda = p_1(\gamma); \gamma \in \Gamma, p_2(\gamma) \in I \}$$

Let $0 < h_1 < h_2 < \ldots$ be an increasing sequence of positive numbers tending to infinity. We set $I_1 = [-h_1, h_1] \subset I_2 = [-h_2, h_2] \subset \ldots$ and we have $\bigcup_1^\infty I_k = \mathbb{R}$. The corresponding sequence of model sets is $\Lambda_k$, $k \in \mathbb{N}$. We have

$$\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_k \subset \ldots$$

and $\bigcup_1^\infty \Lambda_k = p_1(\Gamma)$ is dense in $\mathbb{R}^n$. 

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Now we can define \textit{projective model sets}.

\textbf{Definition 5.2.} Let $a_0 = 0 < a_1 < a_2 < \ldots$ be an increasing sequence of positive numbers tending to infinity. A \textit{projective model set} is defined by

\begin{equation}
\tilde{\Lambda} = \bigcup_{1}^{\infty} \tilde{\Lambda}_k
\end{equation}

where

\begin{equation}
\tilde{\Lambda}_k = \{ \lambda \in \Lambda_k; |\lambda| \geq a_{k-1} \}
\end{equation}

\textbf{Lemma 5.2.} Let $E = \{ k + m\sqrt{2}; (k, m) \in \mathbb{N}^2 \}$. Then $\Lambda = E \cup (-E)$ is a projective model set.

For proving this lemma it suffices to use Definition 3.1 with

$\Gamma = \{ (k + m\sqrt{2}, k - m\sqrt{2}); k, m \in \mathbb{Z} \}$, $h_k = a_k = k$.

A \textit{projective model set} is closed and discrete. A model set is never a \textit{projective model set}. The density of a model set is finite while the density of a \textit{projective model set} is infinite.
Theorem 5.2. Every projective model set $\tilde{\Lambda}$ contains the support of a measure $\mu$ such that

(a) $\mu$ is not a generalized Dirac comb

(b) the Fourier transform $\hat{\mu}$ of $\mu$ is also supported by a projective model set $S$.

6 Construction of $\mu$

For the sake of simplicity the proof will be written in the one dimensional case. The sequences $a_k$, $k \geq 0$, and $h_k$, $k \geq 1$, are defined as above. A closed set $A$ and an open set $B$ are defined by

$$A = \bigcup_{1}^{\infty} \{(x, y); |x| \geq a_{n-1}, |y| \leq h_n\}$$

$$B = \bigcup_{1}^{\infty} \{(x, y); |x| < a_n, |y| > h_n\}$$
Similarly $A^*$ and $B^*$ are defined by two other increasing sequences $a^*_n$, $n \geq 0$, and $h^*_n$, $n \geq 1$, of positive numbers tending to infinity.

Then $\mathbb{R}^2 = A \cup B$ and, similarly, $\mathbb{R}^2 = A^* \cup B^*$.

The first three sequences $a_n, h_n,$ and $a^*_n$ are arbitrary but the sequence $h^*_n$ will be computed by induction.

**Definition 6.1.** With the preceding notations the projective model set defined by $A$ is:

\begin{align*}
(39) \quad \Lambda &= \{p_1(\gamma); \gamma \in \Gamma \cap A\} \\

\text{The conjugate projective model set } Q \text{ is:} \\
(40) \quad Q &= \{p_2(\gamma); \gamma \in \Gamma \cap B\}
\end{align*}

In the Fourier transform side one defines two projective model sets by:

\begin{align*}
(41) \quad S &= \{p_1^*(\gamma^*); \gamma^* \in \Gamma^* \cap A^*\}
\end{align*}

and
\( Z = \{p_2^*(\gamma^*); \gamma^* \in \Gamma^* \cap B^*\} \)

Theorem 3.1 is an easy consequence of the following lemma

**Lemma 6.1.** Given \( a_n, h_n, \) and \( a_n^* \) one can construct a sequence \( h_n^* \) such that there exists a function \( \phi \in S(\mathbb{R}) \) which is not identically 0 and such that

\( \phi = 0 \text{ on } Z, \hat{\phi} = 0 \text{ on } Q \)

Then \( \mu \) is defined by

\[
\mu = \sum_{(x,y) \in \Gamma} \hat{\phi}(y) \delta_x = \sum_{(x,y) \in \Gamma \cap A} \hat{\phi}(y) \delta_x = \sum_{\lambda \in \Lambda} \phi(\lambda) \delta_{\lambda}
\]

where \( (\lambda, \lambda) \in \Gamma \). This implies

\[
\hat{\mu} = \sum_{(u,v) \in \Gamma^*} \phi(v) \delta_u = \sum_{(u,v) \in \Gamma^* \cap A^*} \phi(v) \delta_u = \sum_{u \in S} \phi(\bar{u}) \delta_u
\]

**Theorem 6.1.** Let \( \phi \) be defined by Lemma 4.1. Then \( \mu \) defined by (16) is supported by the closed discrete
set $\Lambda$ defined by (7). Moreover the Fourier transform $\hat{\mu}$ of $\mu$ is supported by the closed discrete set $S$ defined by (13).

This is obvious by construction and it remains to prove Lemma 4.1. To this end let us define:

(46) \[ X_n = \{ p_2(\gamma^*); \gamma^* \in \Gamma^*, |p_1(\gamma^*)| < a_n^* \} \]

(47) \[ B_n^* = \bigcup_{1}^{n} \{ |x| < a_j^*; |y| > h_j^* \} \]

(48) \[ Z_n = \{ p_2^*(\gamma^*); \gamma^* \in \Gamma^* \cap B_n^* \} \subset X_n \]

One can easily construct an increasing sequence $\Omega_n$ consisting of finite union of compact intervals such that

(21) \[ \Omega_n = -\Omega_n, \Omega_n \subset \mathbb{R} \setminus \mathbb{Q}, mes \Omega_n > densX_n \]

Then one starts with $\phi_0 \in \mathcal{S}(\mathbb{R})$ such that $\text{spec}(\phi_0) \subset \Omega_0, \phi_0(0) = 1$ and one inductively constructs a sequence.
of pairs \((\phi_n, h_n^*)\) such that:

\[
\phi_n \in \mathcal{S}(\mathbb{R}) \\
\phi_n(0) = 1 \\
\text{spec}(\phi_n) \subset \Omega_n \\
\|\phi_n - \phi_{n-1}\|_{(m,k)} < 2^{-n}, \quad 0 \leq m, k \leq n \\
\phi_n = 0 \text{ sur } Z_n
\]

Here and in what follows

\[
\|f\|_{(m,k)} = \sup_{x \in \mathbb{R}} |x^m f(k)(x)|
\]

Then \(\phi_n\) tends to a function \(\phi\) in \(\mathcal{S}(\mathbb{R})\) and \(\phi\) has the required properties.

We now construct the pairs \((\phi_n, h_n^*)\).

The construction depends on interpolation on \textit{model sets}. Here is this result.
Let $\Omega \subset \mathbb{R}$ be a compact set. The Paley-Wiener space $PW_{\Omega}$ consists of all functions $f \in L^2(\mathbb{R})$ whose Fourier transform is supported by $\Omega$. A closed discrete set $\Lambda \subset \mathbb{R}$ is a set of stable interpolation for $PW_{\Omega}$ if, for every sequence $c(\lambda) \in l^2(\Lambda)$, there exists an $f \in PW_{\Omega}$ such that $c(\lambda) = f(\lambda)$, $\lambda \in \Lambda$. Model sets have the following property [5]:

**Lemma 6.2.** Let $\Omega$ be any finite union of intervals. If the measure of $\Omega$ is larger than the density of the model set $\Lambda$, then $\Lambda$ is a set of stable interpolation for $PW_{\Omega}$.

Stable interpolation in $L^2$ implies interpolation in the Schwartz class.

**Lemma 6.3.** Let $\Lambda$ be a set of stable interpolation for $PW_{\Omega}$. Then for every pair of integers $k, m \in \mathbb{N}$ and for every $\epsilon > 0$, there exists a constant $C_{m,k}(\Lambda, \Omega, \epsilon)$ such that for every sequence $c(\lambda)$, $\lambda \in \Lambda$, fulfilling

$$|c(\lambda)| \leq (1 + |\lambda|)^{-m}, \lambda \in \Lambda,$$

there exists a function $f \in PW_{\Omega+[-\epsilon,\epsilon]} \cap S(\mathbb{R})$, such that $c(\lambda) = f(\lambda)$, $\lambda \in \Lambda$, and

$$\|f\|_{(m,k)} \leq C_{m,k}(\Lambda, \Omega, \epsilon).$$
We are ready for constructing $\phi_n$. One defines $\phi_{n+1}$ by $\phi_{n+1} = \phi_n - f_n$ where $f_n$ is defined as follows. One denotes by $J$ a finite union of intervals such that

$$J + [-\epsilon, \epsilon] \subset \Omega_{n+1}, \quad \text{mes } J > \frac{2a_{n+1}^*}{\text{vol } \Gamma^*}$$

Lemma 4.3 will be applied to the model set $X_{n+1}$ defined by (18) and to $m \in [0, n+1]$, $k \in [0, n+1]$. Observe that $X_{n+1}$ does not depend on $h_{n+1}^*$. This yields a constant $C = C_{n+1}(\Lambda, J, \epsilon)$. Then one fixes $h_{n+1}^*$ large enough to obtain

$$|\lambda| > h_{n+1}^*, \quad \Rightarrow \quad |\phi_n(\lambda)|(1+|\lambda|)^{n+1} \leq \frac{1}{C \cdot 2^{n+1}}, \quad \lambda \in X_{n+1}$$

Both $B_{n+1}^*$ and $Z_{n+1}$ are defined by this choice.

We define $c(\lambda)$, $\lambda \in X_{n+1}$, by $c(\lambda) = 0$ if $|\lambda| \leq h_{n+1}^*$ and $c(\lambda) = \phi_n(\lambda)$ if $|\lambda| > h_{n+1}^*$.

Lemma 4.3 yields a function $f_n$ in the Schwartz class such that $f_n(\lambda) = c(\lambda)$, $\lambda \in X_{n+1}$ with $\|f_n\|_{(m,k)} \leq 2^{-n-1}$, $0 \leq m, k \leq n + 1$. One sets $\phi_{n+1} = \phi_n - f_n$. 

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Lemma 6.4. The above construction implies $\phi_{n+1} = 0$ on $Z_{n+1}$.

If $|\lambda| \leq h_{n+1}^*$ one uses the following three properties $Z_{n+1} \cap [-h_{n+1}^*, h_{n+1}^*] \subset Z_n$, $f_n(\lambda) = 0$ on $X_{n+1} \cap [-h_{n+1}^*, h_{n+1}^*]$ and $\phi_n = 0$ on $Z_n$. If $|\lambda| > h_{n+1}^*$ we have $f_n = \phi_n$ and $\phi_{n+1} = 0$. Observe that $0 \in X_{n+1}$ which implies $f_n(0) = 0$.

Then $\phi_n$ converges to $\phi$ in the Schwartz class and $\phi$ has the required properties.

References

